Robotic 3D Vision

Lecture 4: Probabilistic State Estimation – Full Posterior Optimization

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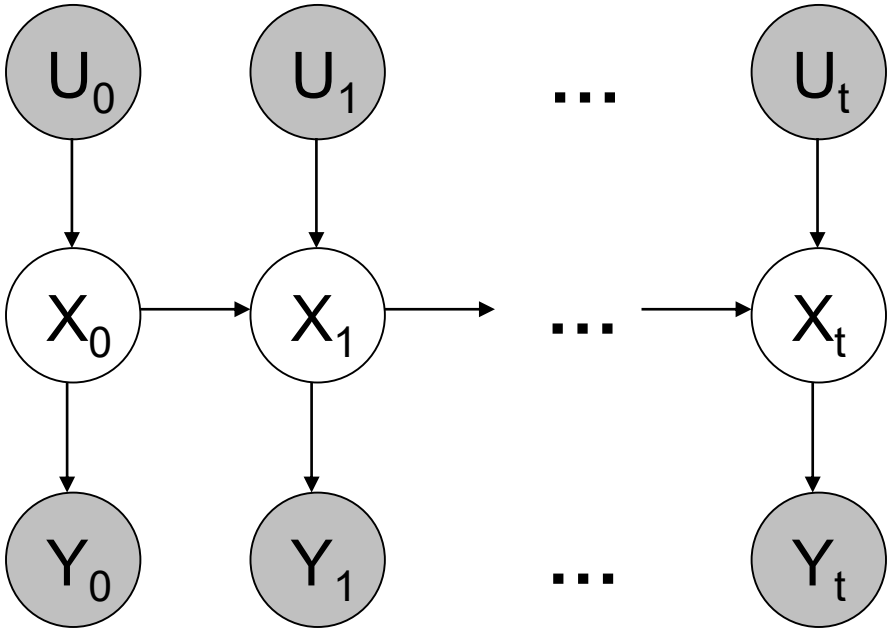
What We Will Cover Today

- Probabilistic modelling of state estimation problems
  - Examples of observation and state-transition models
  - Example: Monte-Carlo Localization
- Short intro to graphical models
  - Directed graphical models and factor graphs
- Full posterior optimization
  - Non-linear least squares
  - Optimization methods
  - Tools and frameworks

Recap: Probabilistic Model of Time-Sequential Processes

- Hidden state X gives rise to noisy observations Y
- At each time t,
  - the state changes stochastically from X_{t-1} to X_t
  - state change depends on action U_t
  - we get a new observation Y_t



Recap: Why Probabilistic State Estimation?

- Probabilistic modelling accounts for uncertainties
- State estimation: Inference in probabilistic model
- Cope with noisy state transitions and observations
- Maintain uncertainty in the state estimate
- Principled approaches to update the state estimate distribution based on probability theory

Recap: Observation and State-Transition Models

- We assume
  - Knowledge about probability distribution of observations

$$p\left(Y_t \middle| X_{0:t}, U_{0:t}, Y_{0:t-1}\right)$$

  - Knowledge about probabilistic dynamics of state transitions

$$p\left(X_t \middle| X_{0:t-1}, U_{0:t}\right)$$

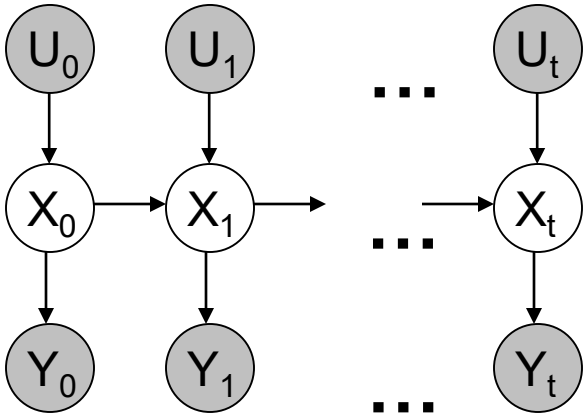
Recap: Markov Assumptions

- Only the immediate past matters for a state transition

$$p\left(X_t \middle| X_{0:t-1}, U_{0:t}\right) = \boxed{p\left(X_t \middle| X_{t-1}, U_t\right)}$$ **state transition model**

- Observations depend only on the current state

$$p\left(Y_t \middle| X_{0:t}, U_{0:t}, Y_{0:t-1}\right) = \boxed{p\left(Y_t \middle| X_t\right)}$$ **observation model**

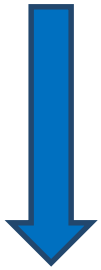


Recap: Predict-Correct Cycle

- Prediction:

$$p(X_t \mid y_{0:t-1}, u_{0:t}) = \int p(X_t \mid X_{t-1}, u_t)\, p(X_{t-1} \mid y_{0:t-1}, u_{0:t-1})\, dX_{t-1}$$

observation y_t



action u_t

- Correction:

$$p(X_t \mid y_0, \ldots, y_t) = \frac{p(y_t \mid X_t)\, p(X_t \mid y_{0:t-1}, u_{0:t})}{\int p(y_t \mid X_t)\, p(X_t \mid y_{0:t-1}, u_{0:t})\, dX_t}$$

Recap: Kalman Filter

- Kalman filters (KFs) instantiate recursive Bayesian filtering for a specific class of state transition and observation models

  - Linear state transition model with Gaussian noise:

$$\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \boldsymbol{\epsilon} \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{d_t})$$

  - Linear observation model with Gaussian noise:

$$\mathbf{z}_t = \mathbf{C}_t \mathbf{x}_t + \boldsymbol{\delta} \qquad \boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{m_t})$$

  - Gaussian initial state estimate: $\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$

Recap: Kalman Filter Prediction & Correction

- Efficient closed-form correction and prediction steps which involve manipulation of Gaussians
- The state estimate can be represented as a Gaussian distribution

$$\mathbf{x}_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$

- Prediction:

$$\boldsymbol{\mu}_t^- = \mathbf{A}_t \boldsymbol{\mu}_{t-1}^+ + \mathbf{B}_t \mathbf{u}_t$$
$$\boldsymbol{\Sigma}_t^- = \mathbf{A}_t \boldsymbol{\Sigma}_{t-1}^+ \mathbf{A}_t^\top + \boldsymbol{\Sigma}_{d_t}$$

- Correction:

$$\mathbf{K}_t = \boldsymbol{\Sigma}_t^- \mathbf{C}_t^\top \left(\mathbf{C}_t \boldsymbol{\Sigma}_t^- \mathbf{C}_t^\top + \boldsymbol{\Sigma}_{m_t} \right)^{-1}$$
$$\boldsymbol{\mu}_t^+ = \boldsymbol{\mu}_t^- + \mathbf{K}_t (\mathbf{y}_t - \mathbf{C}_t \boldsymbol{\mu}_t^-)$$
$$\boldsymbol{\Sigma}_t^+ = (\mathbf{I} - \mathbf{K}_t \mathbf{C}_t) \boldsymbol{\Sigma}_t^-$$

Kalman gain

Recap: Extended Kalman Filter (EKF)

- Non-linear state-transition model with Gaussian noise:

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \boldsymbol{\epsilon}_t \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{d_t})$$

- Non-linear observation model with Gaussian noise:

$$\mathbf{y}_t = h(\mathbf{x}_t) + \boldsymbol{\delta}_t \qquad \boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{m_t})$$

- How to cope with non-linear system?

- Idea: linearize the models in each time step

$$\Rightarrow \quad \mathbf{x}_t \approx g(\mathbf{x}^0_{t-1}, \mathbf{u}_t) + \nabla g(\mathbf{x}, \mathbf{u}_t)|_{\mathbf{x}=\mathbf{x}^0_{t-1}} \left(\mathbf{x}_{t-1} - \mathbf{x}^0_{t-1}\right) + \boldsymbol{\epsilon}_t$$

$$\Rightarrow \quad \mathbf{y}_t \approx h(\mathbf{x}^0_t) + \nabla h(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^0_t} \left(\mathbf{x}_t - \mathbf{x}^0_t\right) + \boldsymbol{\delta}_t$$

Recap: EKF Prediction & Correction

- Efficient approximate correction and prediction steps which involve manipulation of Gaussians and linearization
- The state estimate can be represented as a Gaussian distribution

$$\mathbf{x}_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$

- Prediction:

$$\boldsymbol{\mu}_t^- = g(\boldsymbol{\mu}_{t-1}^+, \mathbf{u}_t)$$

$$\boldsymbol{\Sigma}_t^- = \mathbf{G}_t \boldsymbol{\Sigma}_{t-1}^+ \mathbf{G}_t^\top + \boldsymbol{\Sigma}_{d_t} \qquad \mathbf{G}_t := \nabla g(\mathbf{x}, \mathbf{u}_t)|_{\mathbf{x}=\boldsymbol{\mu}_{t-1}^+}$$

- Correction:

$$\mathbf{K}_t = \boldsymbol{\Sigma}_t^- \mathbf{H}_t^\top \left(\mathbf{H}_t \boldsymbol{\Sigma}_t^- \mathbf{H}_t^\top + \boldsymbol{\Sigma}_{m_t}\right)^{-1}$$

$$\boldsymbol{\mu}_t^+ = \boldsymbol{\mu}_t^- + \mathbf{K}_t \left(\mathbf{y}_t - h(\boldsymbol{\mu}_t^-)\right) \qquad \mathbf{H}_t := \nabla h(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\mu}_t^-}$$

$$\boldsymbol{\Sigma}_t^+ = \left(\mathbf{I} - \mathbf{K}_t \mathbf{H}_t\right) \boldsymbol{\Sigma}_t^-$$

Recap: Particle Filter (PF)

- Non-linear observation and state-transition distributions

$$p(\mathbf{y}_t \mid \mathbf{x}_t) \qquad p(\mathbf{x}_t \mid \mathbf{x}_{t-1}, \mathbf{u}_t)$$

- State estimate (full posterior!) represented as a set of weighted samples

$$\left\{\mathbf{x}_{0:t}^i, w_t^i\right\}_{i=1}^N$$

$$p(\mathbf{x}_{0:t} \mid \mathbf{y}_{0:t}, \mathbf{u}_{1:t}) \approx \sum_{i=1}^N w_t^i \delta_{\mathbf{x}_{0:t}^i}(\mathbf{x}_{0:t})$$

- The weighted samples a.k.a. particles are propagated and updated over time to approximate the full posterior

Recap: Sampling Importance Resampling (SIR)

- Sequential update:
  - Particle update: $\mathbf{x}_t^i \sim q(\mathbf{x}_t \mid \mathbf{x}_{t-1}^i, \mathbf{y}_t, \mathbf{u}_t)$
  - Weight update: $w_t^i = w_{t-1}^i \frac{p(\mathbf{y}_t|\mathbf{x}_t^i)\, p(\mathbf{x}_t^i|\mathbf{x}_{t-1}^i,\mathbf{u}_t)}{q(\mathbf{x}_t|\mathbf{x}_{t-1}^i,\mathbf{y}_t,\mathbf{u}_t)}$
  - Resampling: Draw new particles with replacement with probability proportional to weights

SIR Algorithm

- At each time step t:

$\eta = 0$

for $i = 1{:}N$

$\mathbf{x}_t^i \sim q(\mathbf{x}_t \mid \mathbf{x}_{t-1}^i, \mathbf{y}_t, \mathbf{u}_t)$

$w_t^i = w_{t-1}^i \frac{p(\mathbf{y}_t|\mathbf{x}_t^i)\, p(\mathbf{x}_t^i|\mathbf{x}_{t-1}^i, \mathbf{u}_t)}{q(\mathbf{x}_t|\mathbf{x}_{t-1}^i, \mathbf{y}_t, \mathbf{u}_t)}$

$\eta = \eta + w_t^i$

end

for $i = 1{:}N$

$w_t^i = w_t^i / \eta$

end

if $N_{eff} <$ threshold

  for $i = 1 : N$

    randomly draw $\overline{\mathbf{x}}_t^i$ with replacement with probability $p(\overline{\mathbf{x}}_t^i = \mathbf{x}_t^i) \propto w_t^i$

  end

  for $i = 1 : N$

    $\mathbf{x}_t^i \leftarrow \overline{\mathbf{x}}_t^i$

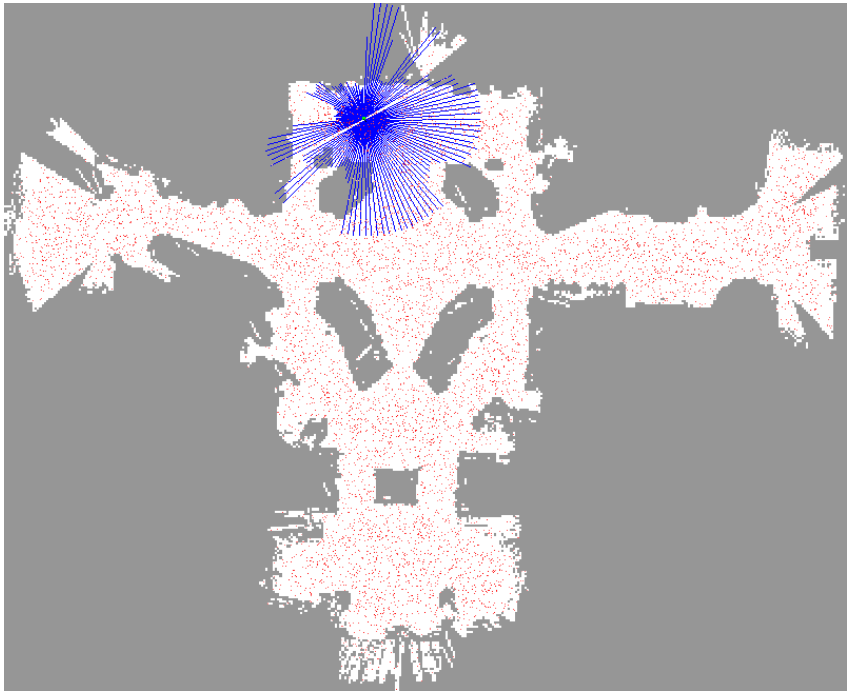
    $w_t^i \leftarrow 1$

  end

end

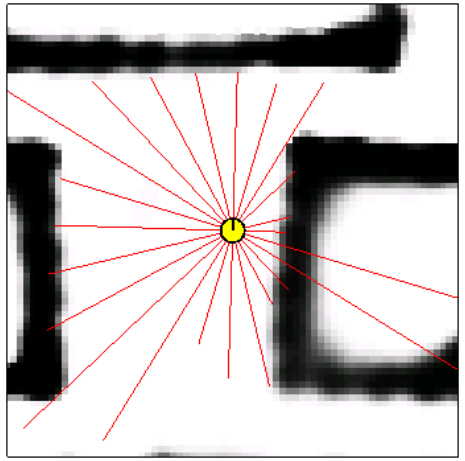
Example: Laser-based Monte Carlo Localization

- Where is the robot (position and orientation)?
- Indoor environment
- Robot moves on flat ground (2D plane)
- Laser sensor measures distance to obstacles in a 2D plane
- Known map

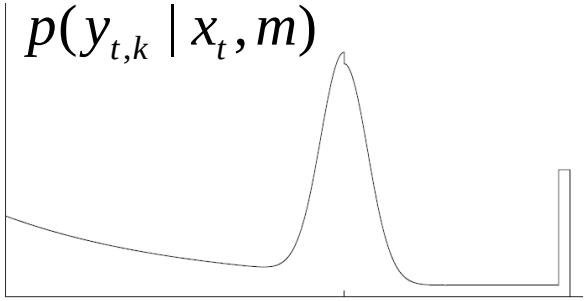


Example Observation Model

- Measurement can be due …
  - a known obstacle
  - an unexpected obstacle (people, furniture, …)
  - missing all obstacles (total reflection, glass, …).
- Noise is due to uncertainty …
  - in measuring distance to known obstacle.
  - in position of known obstacles.
  - in position of additional obstacles.
  - whether obstacle is missed.

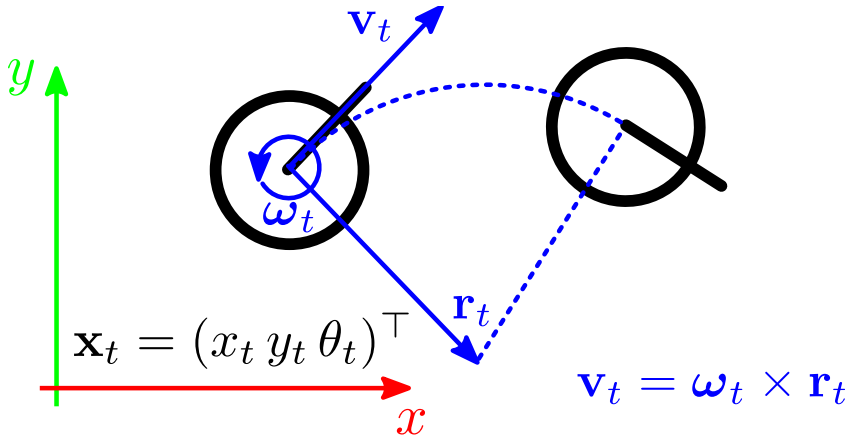


$$p(y_t \mid x_t, m) = \prod_{k=1}^{K} p(y_{t,k} \mid x_t, m)$$



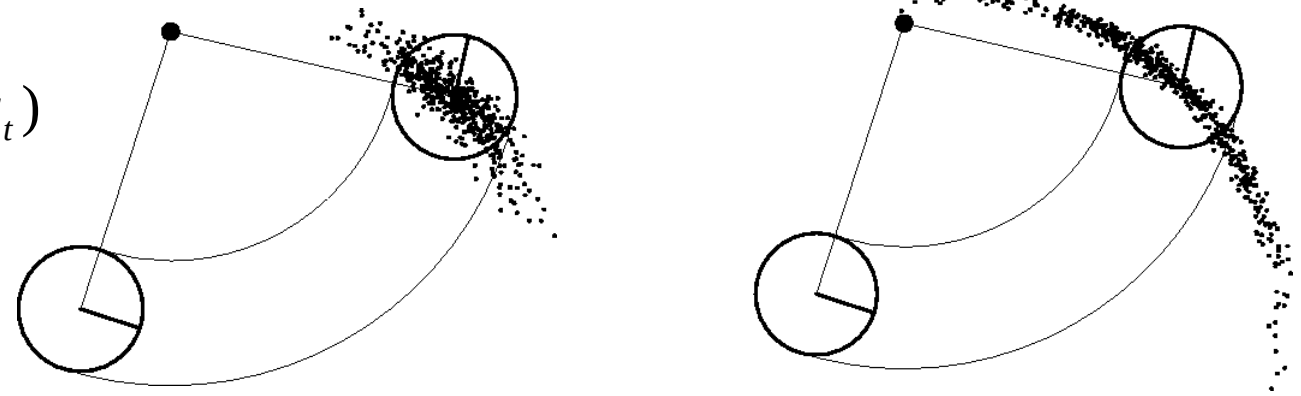
Example State-Transition Model

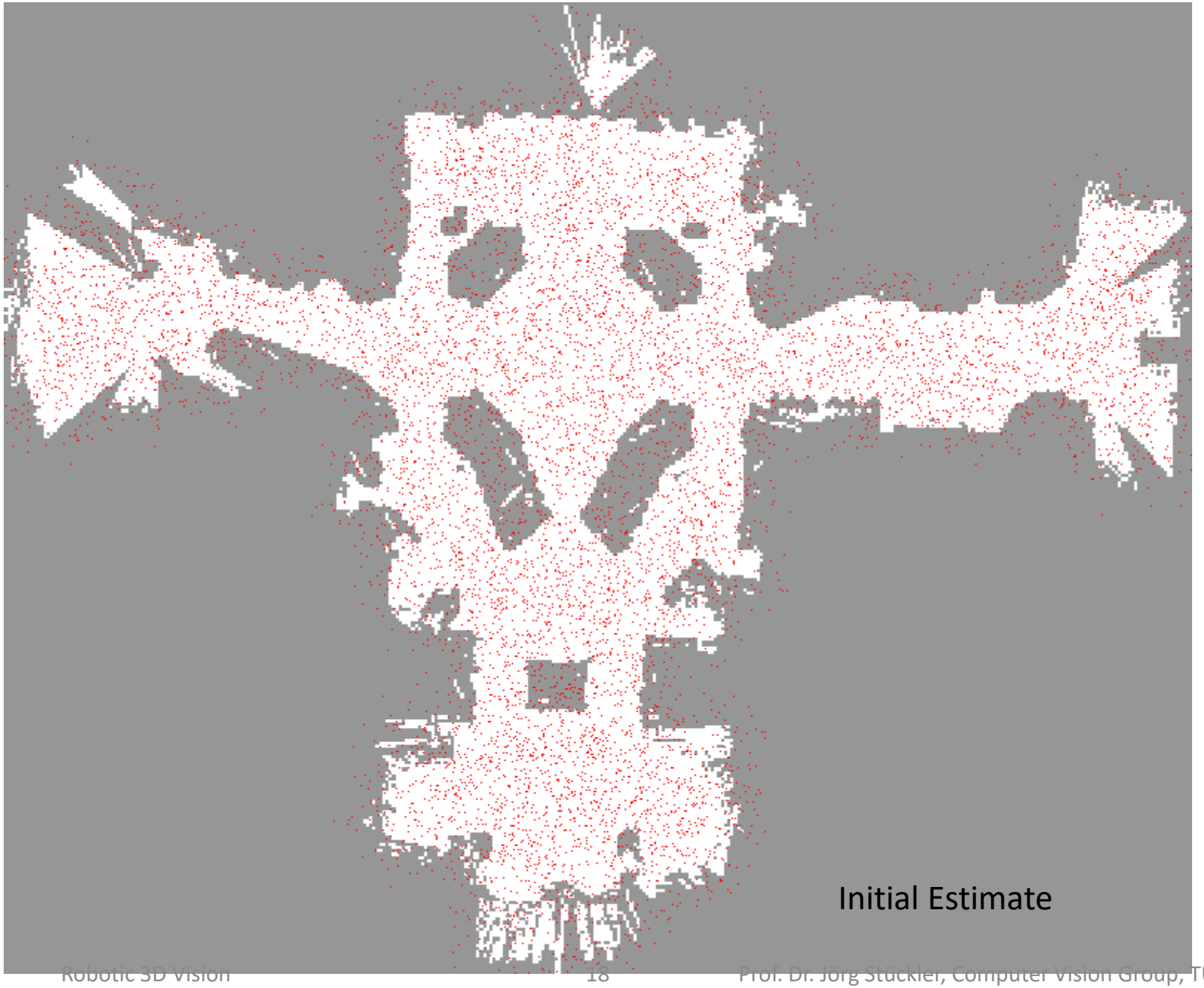
- Velocity-based motion model of a robot in the 2D plane
- Robot actions:
  - Linear velocity $\mathbf{v}_t$
  - Rotational velocity $\boldsymbol{\omega}_t$

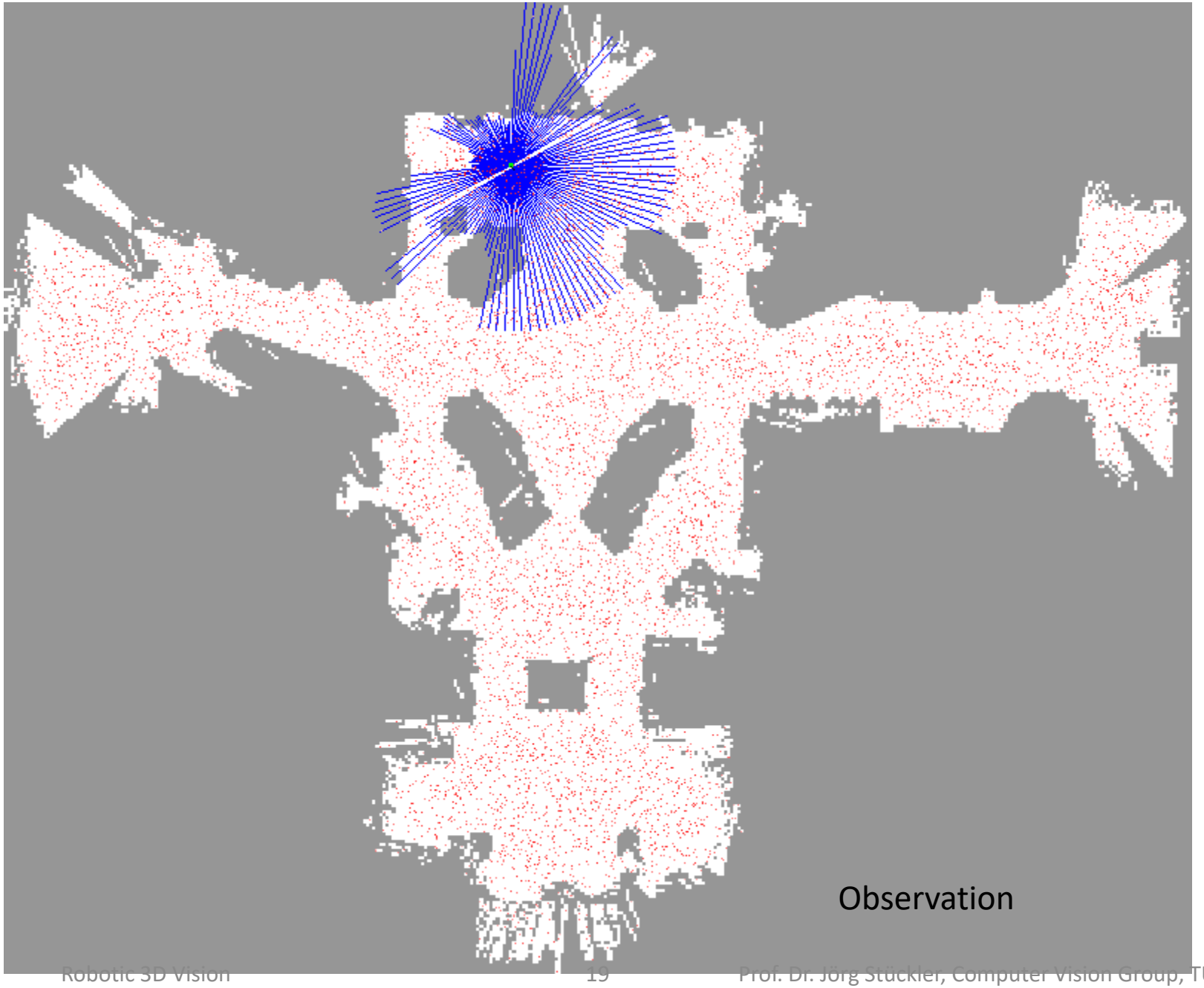


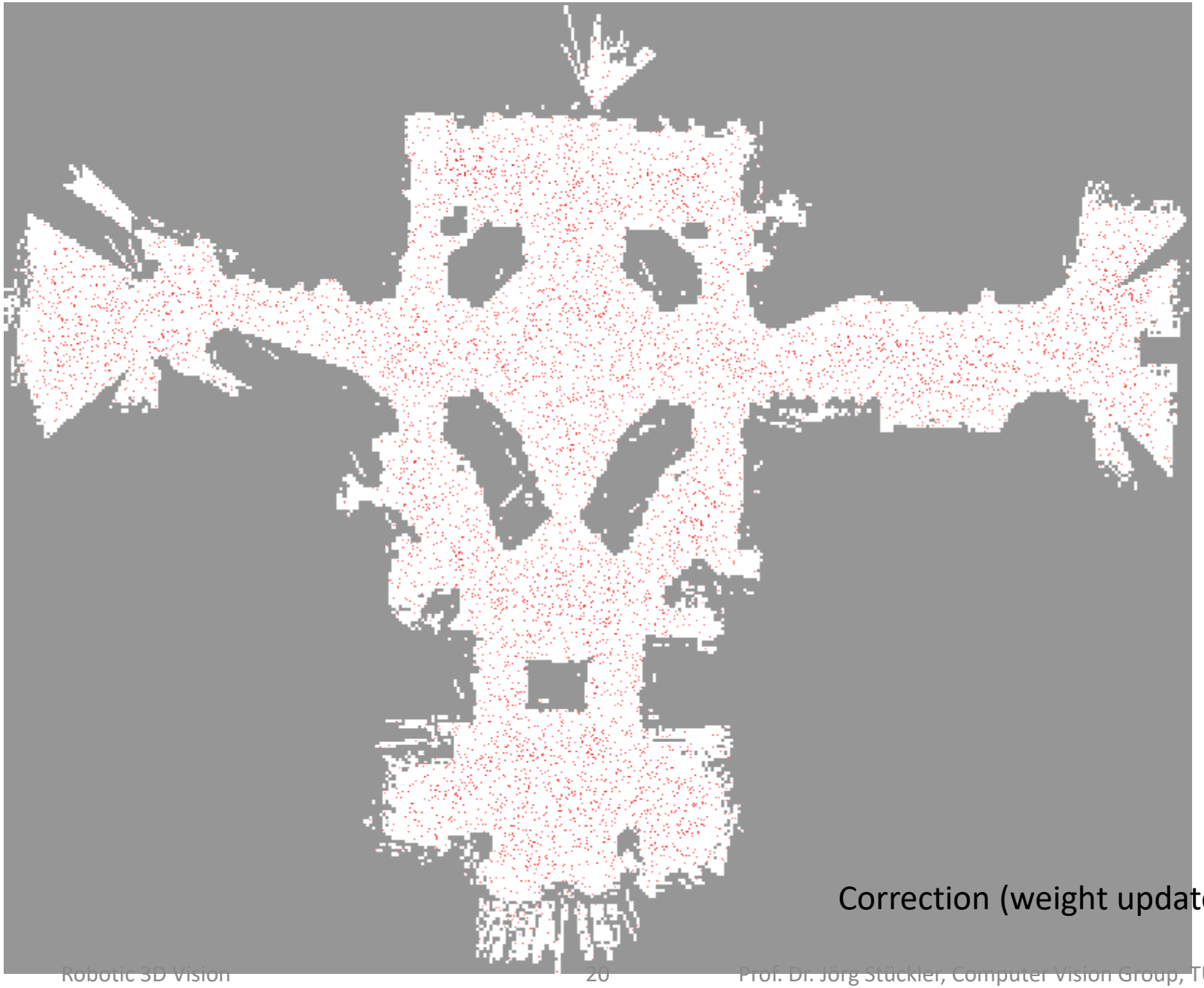
- Actions are executed with uncertainty (f.e. Gaussian noise on velocities)

$p(x_t \mid x_{t-1}, u_t)$

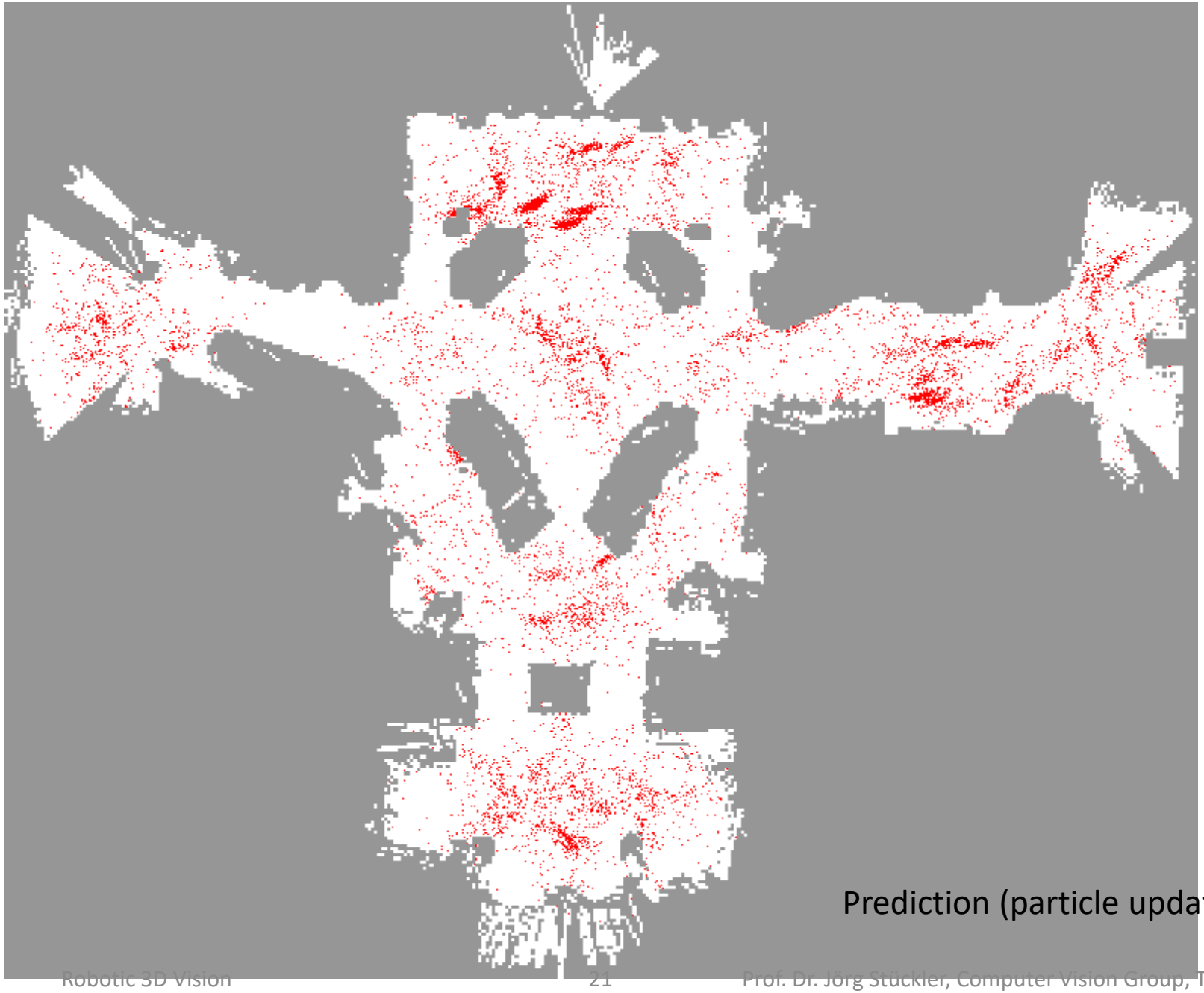



Initial Estimate

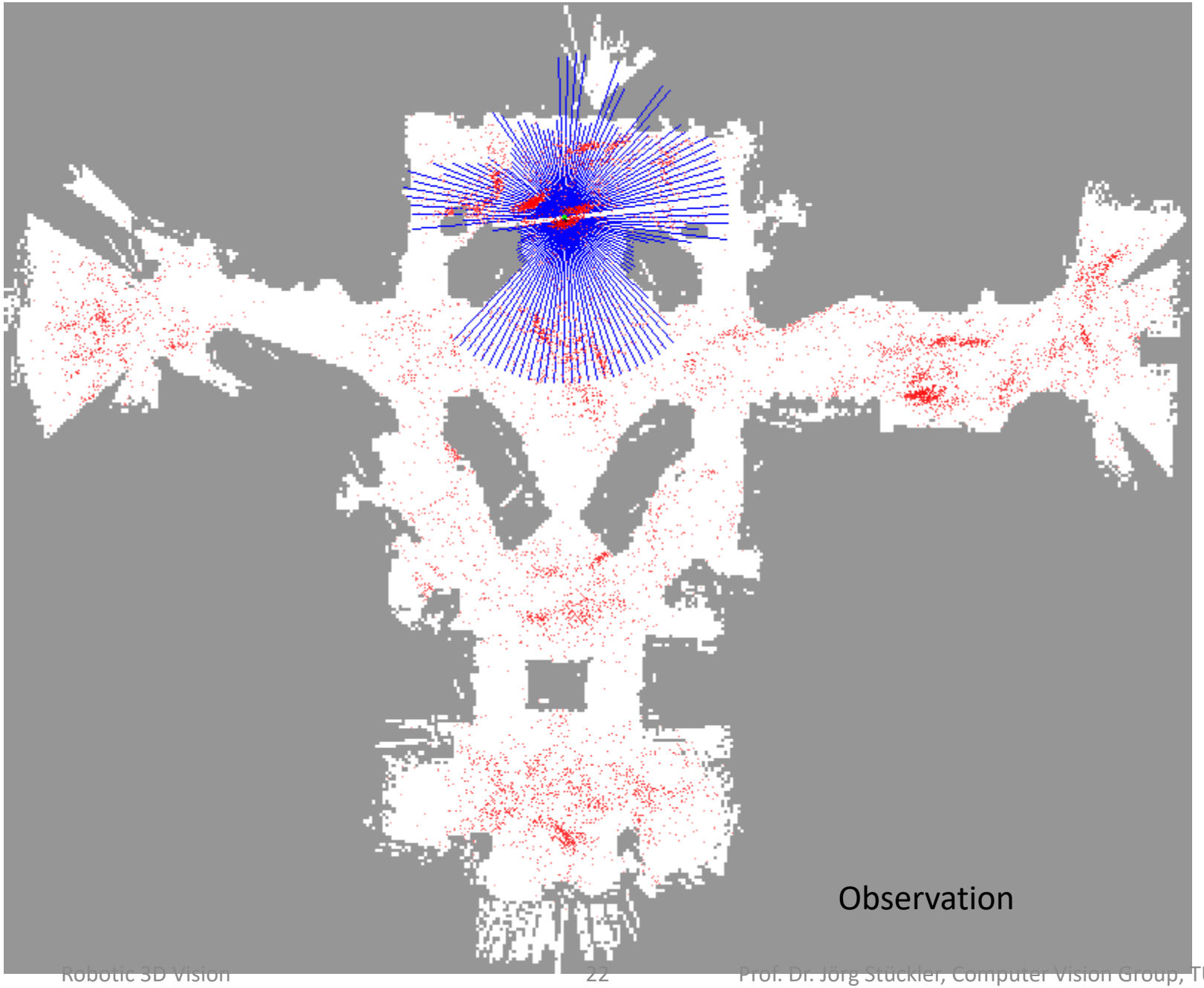

Observation



Correction (weight update)



Prediction (particle update)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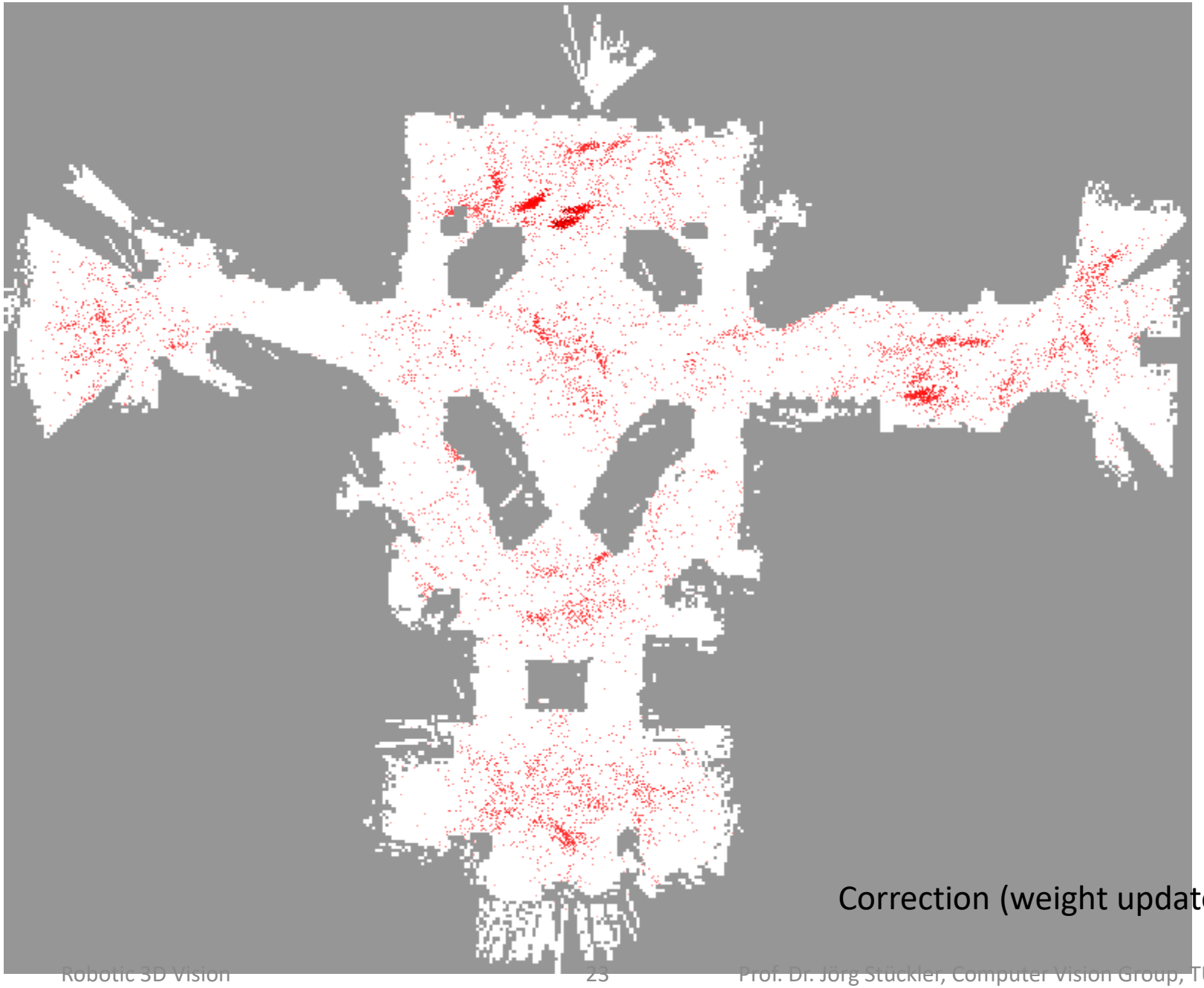
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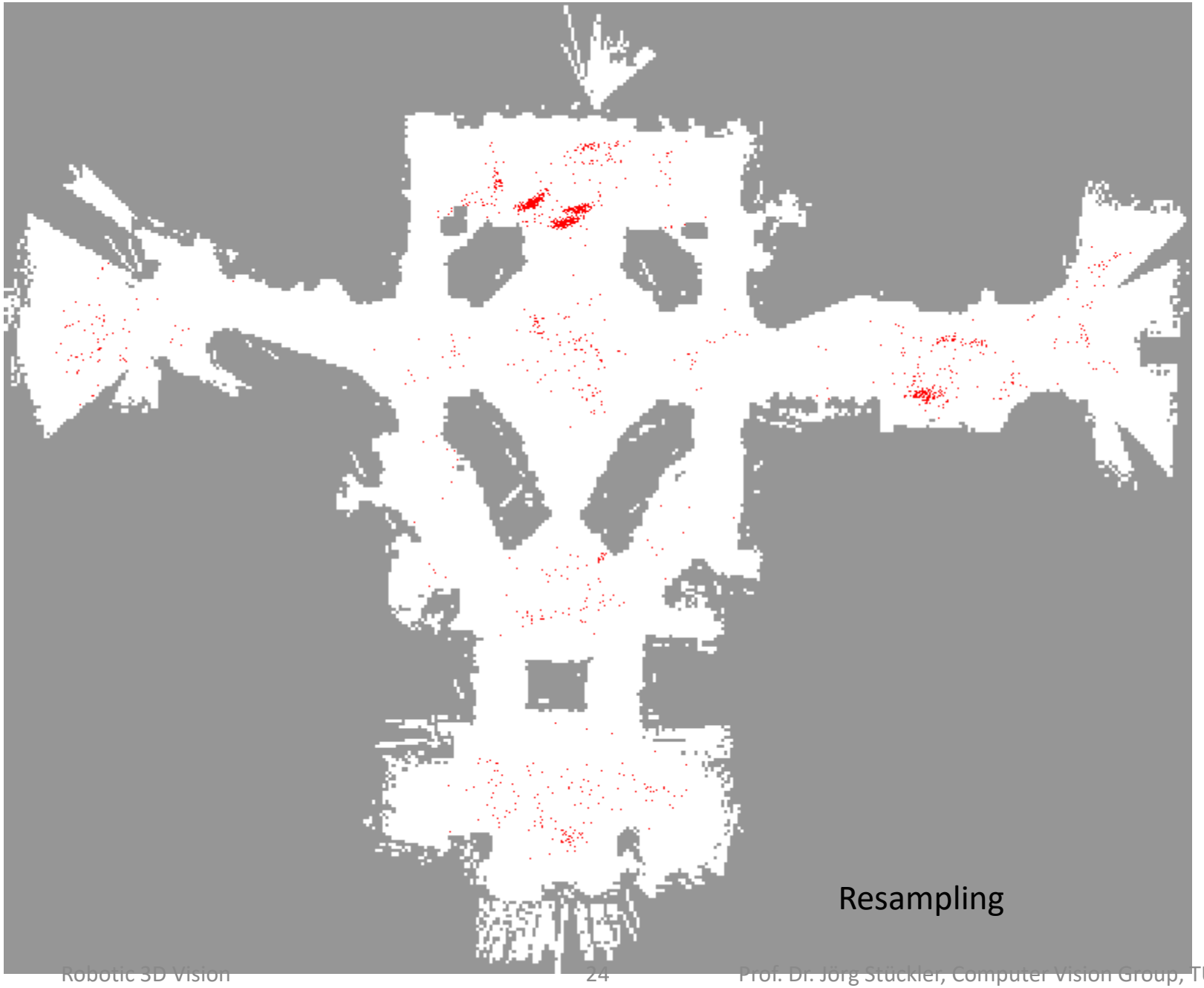
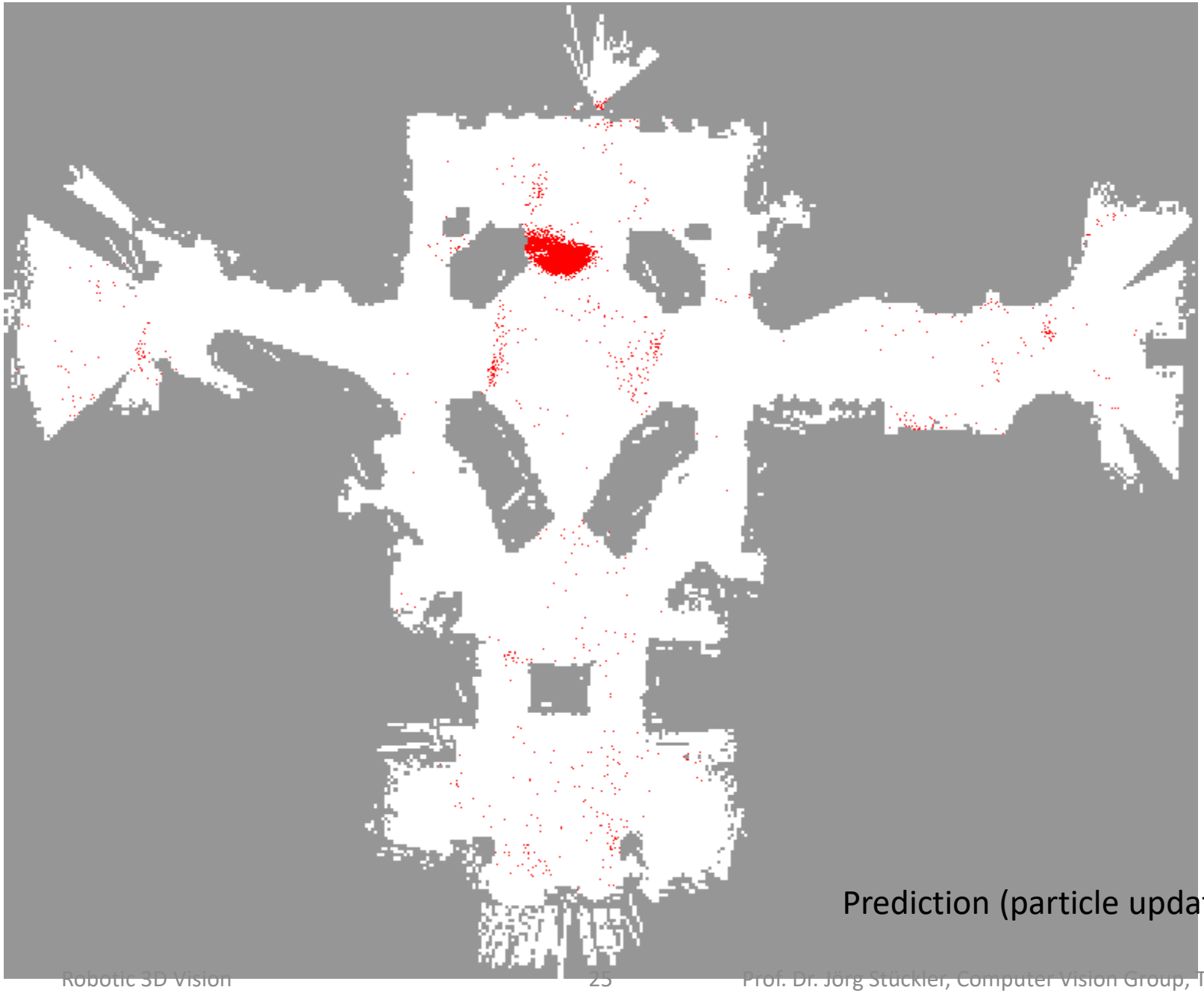

Resampling



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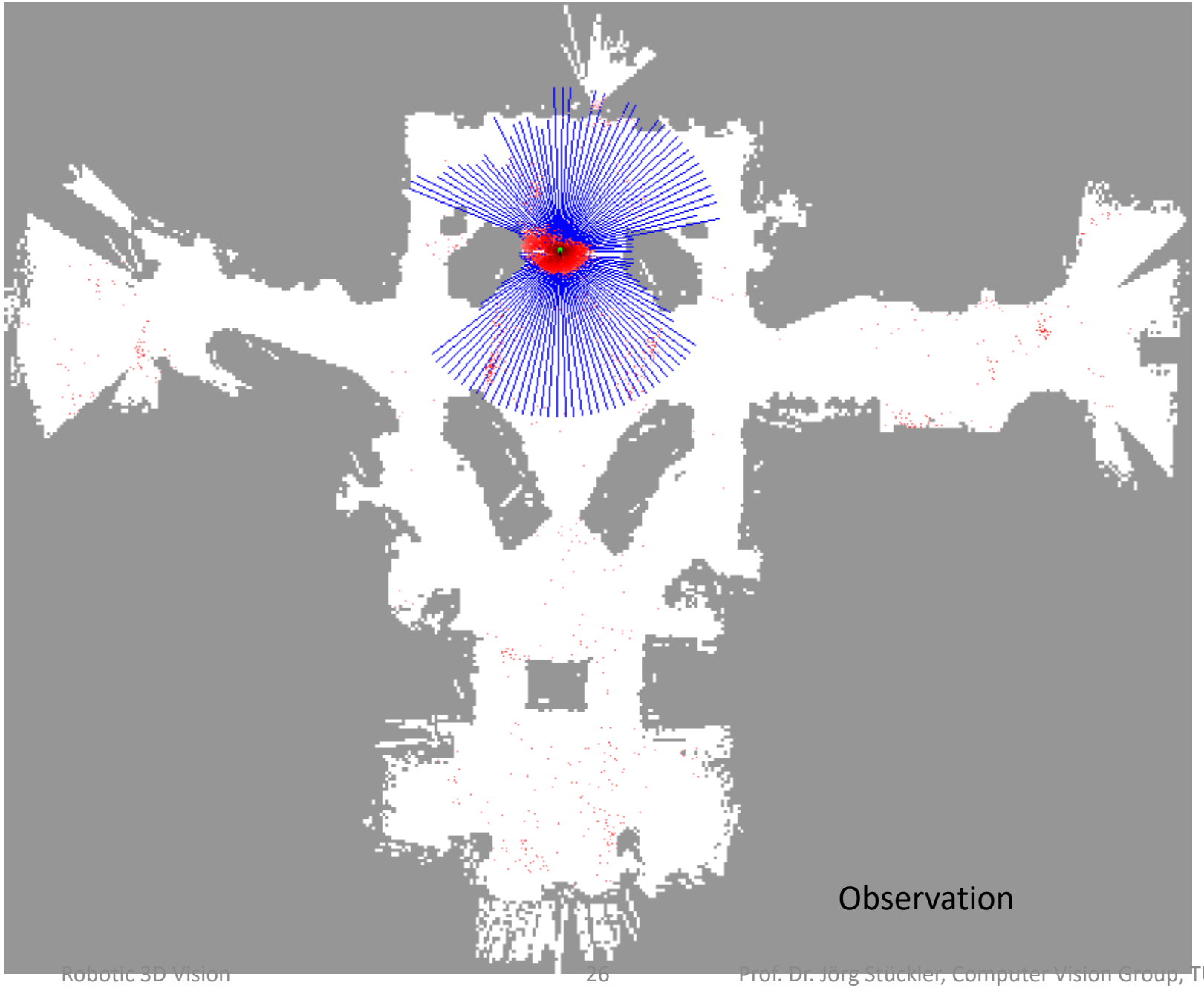
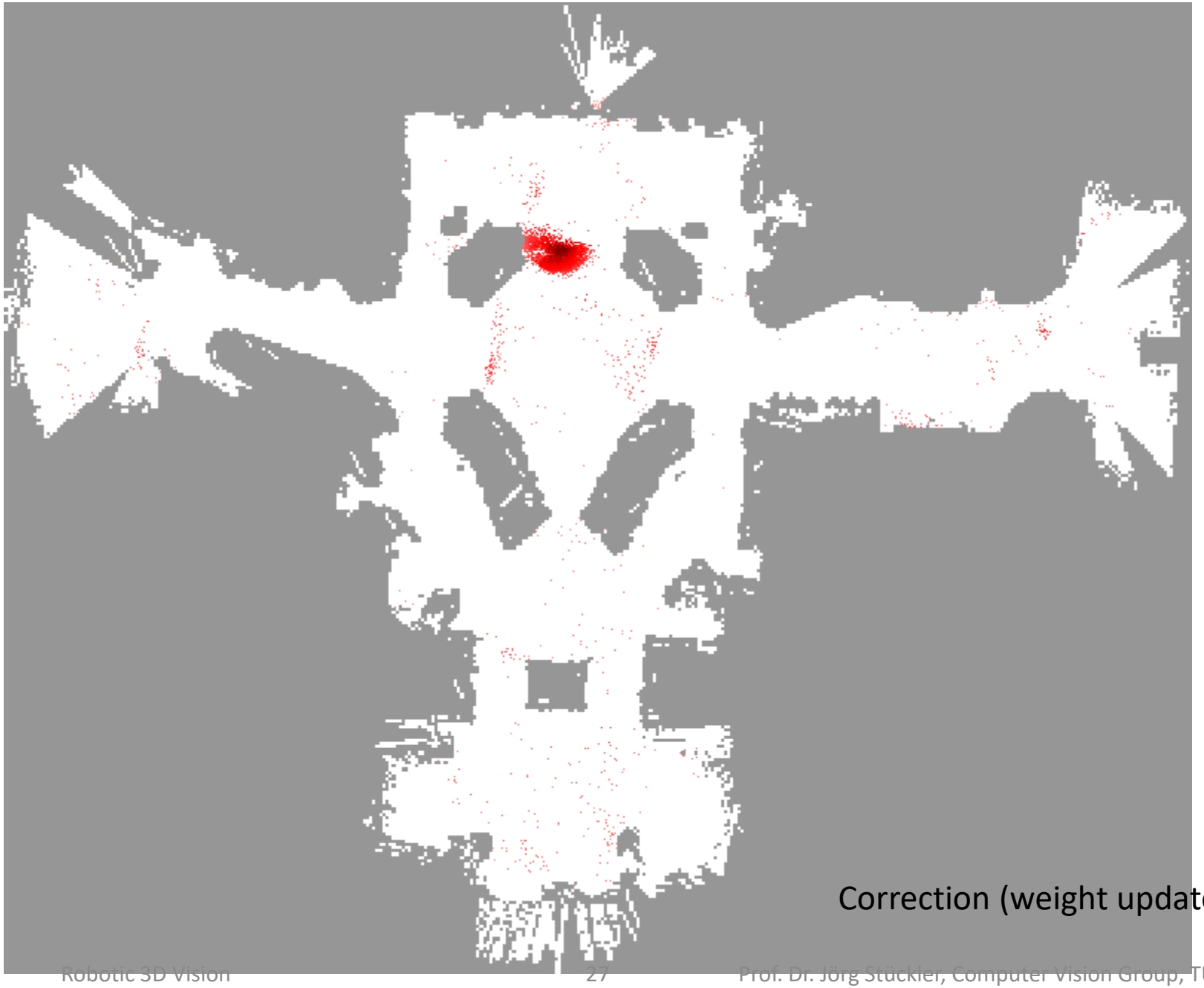

Observation



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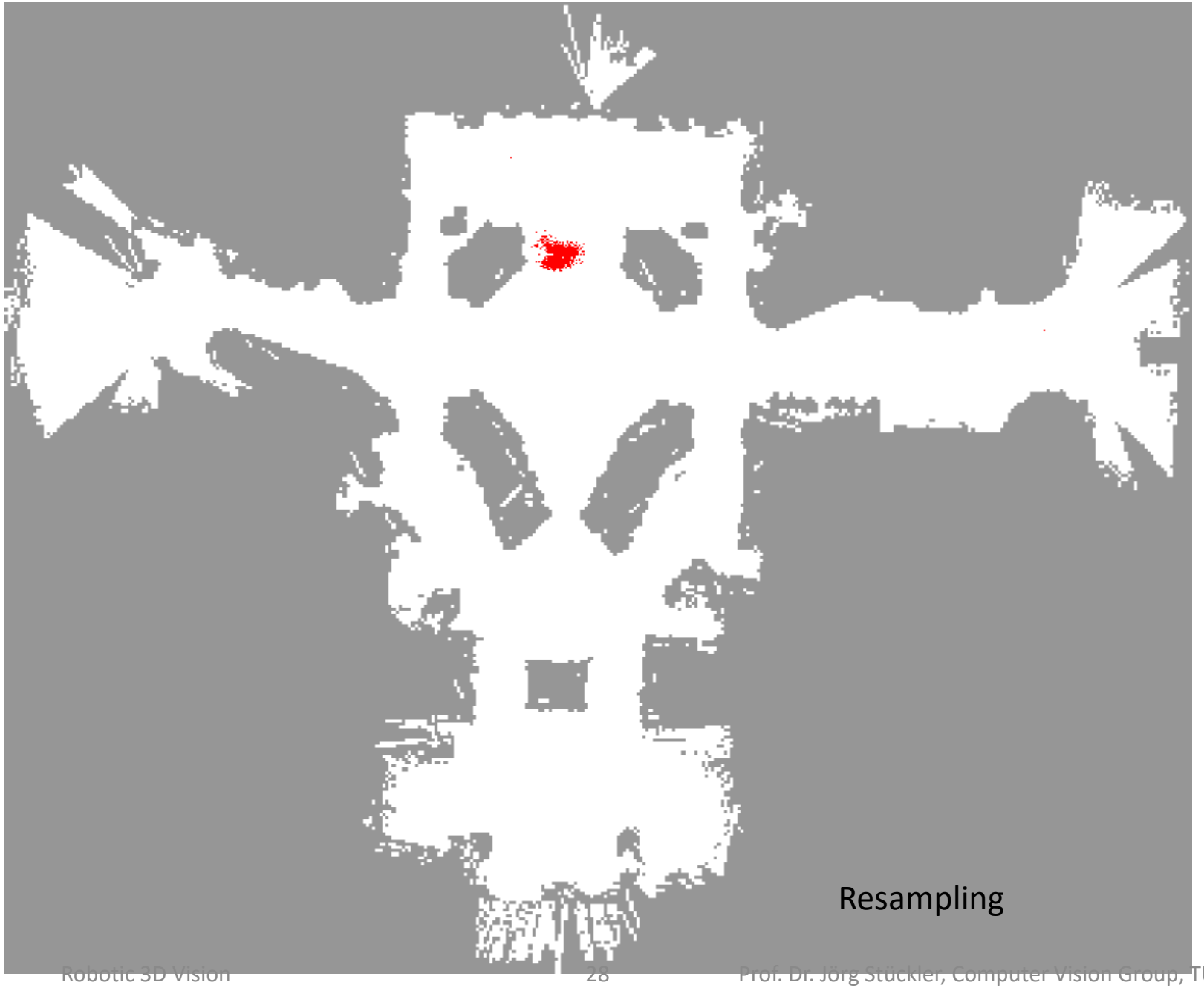
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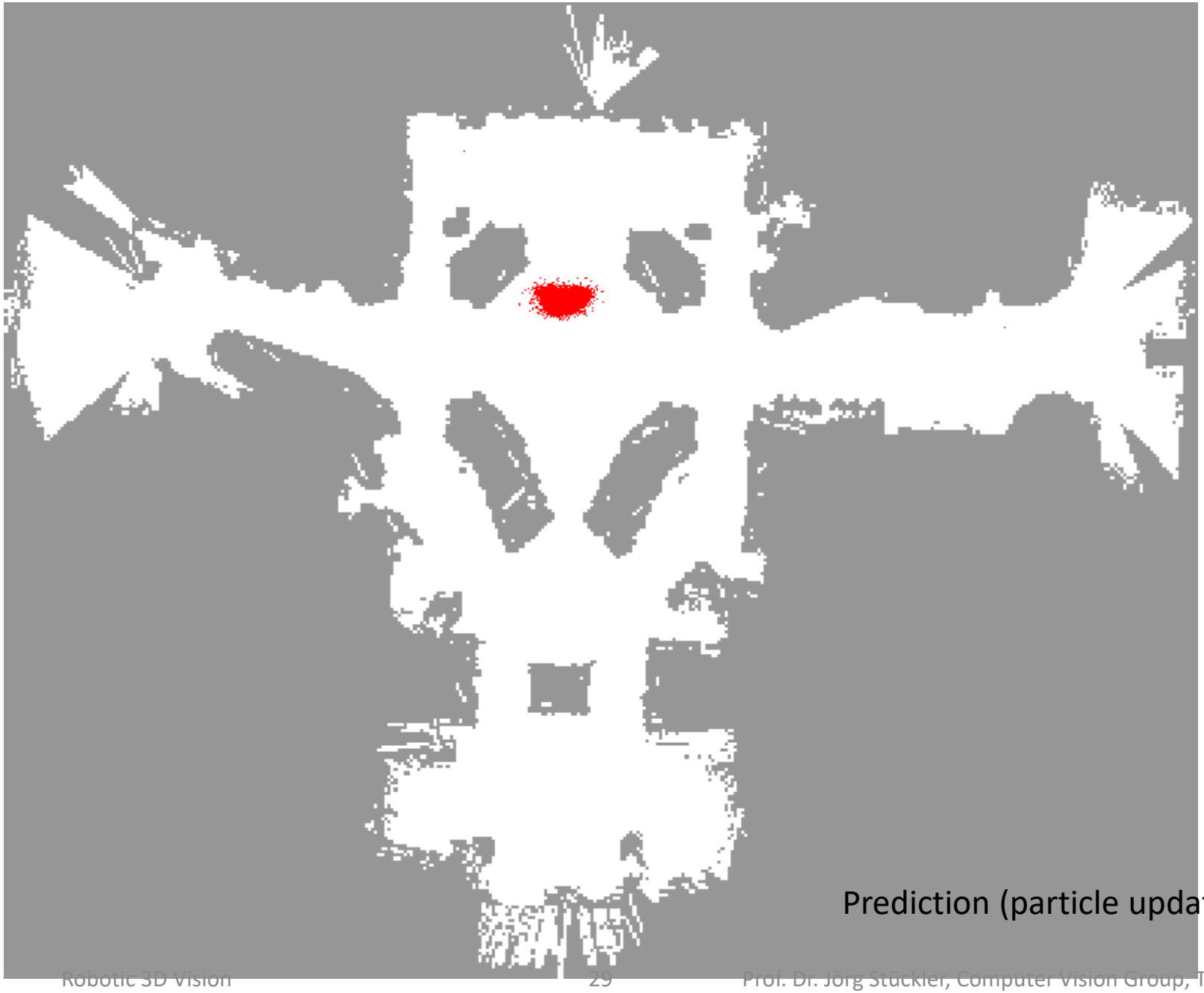
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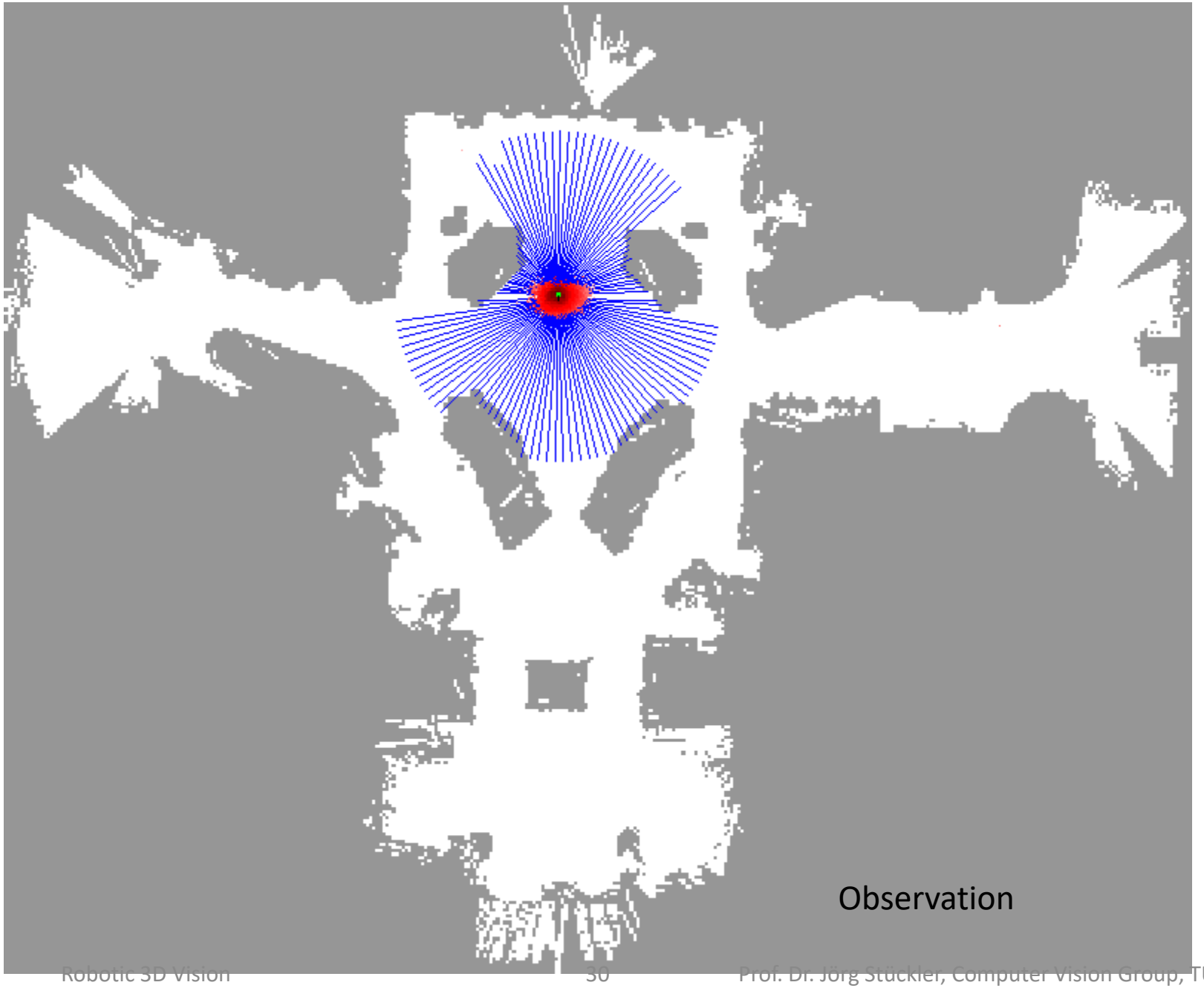
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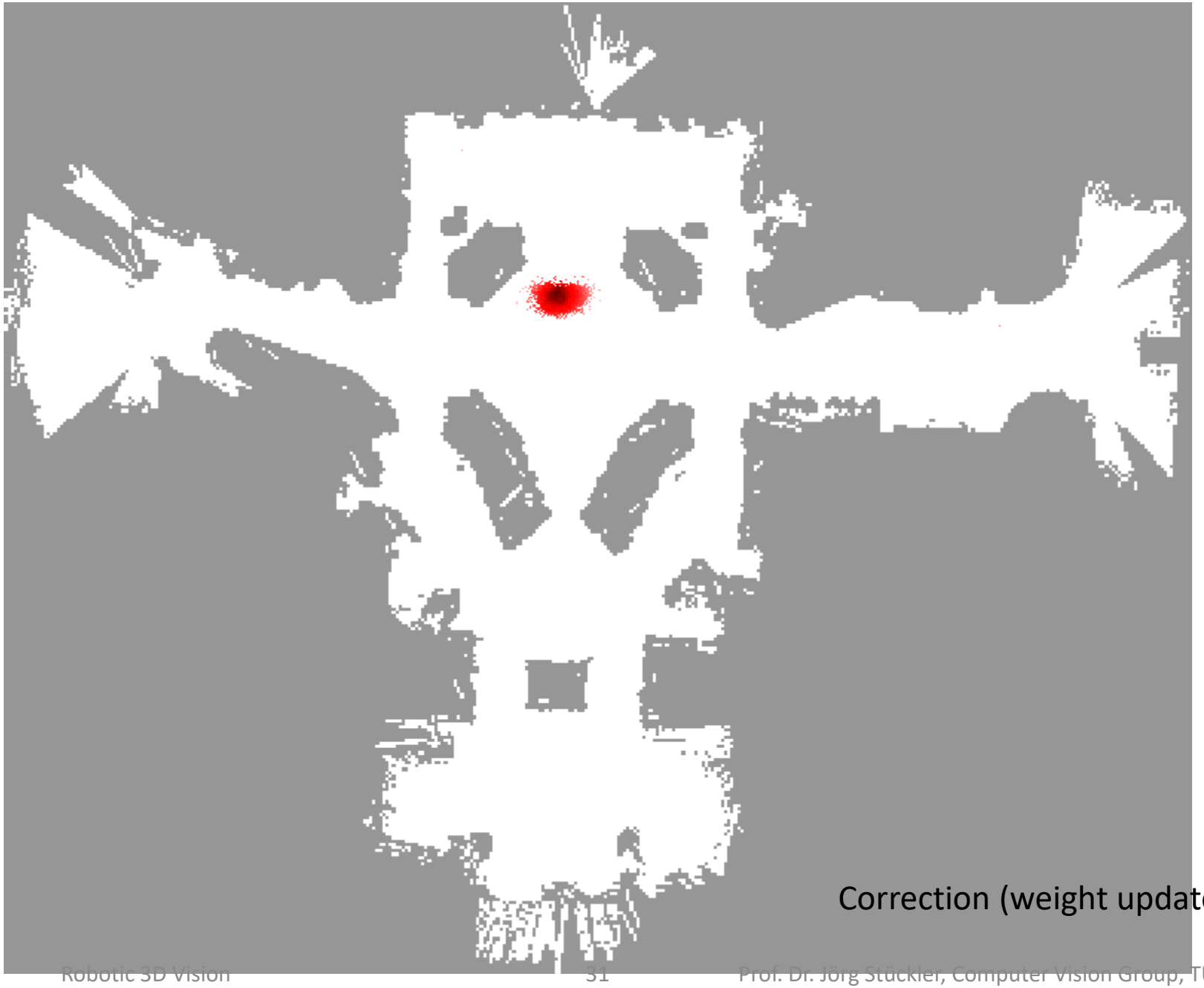

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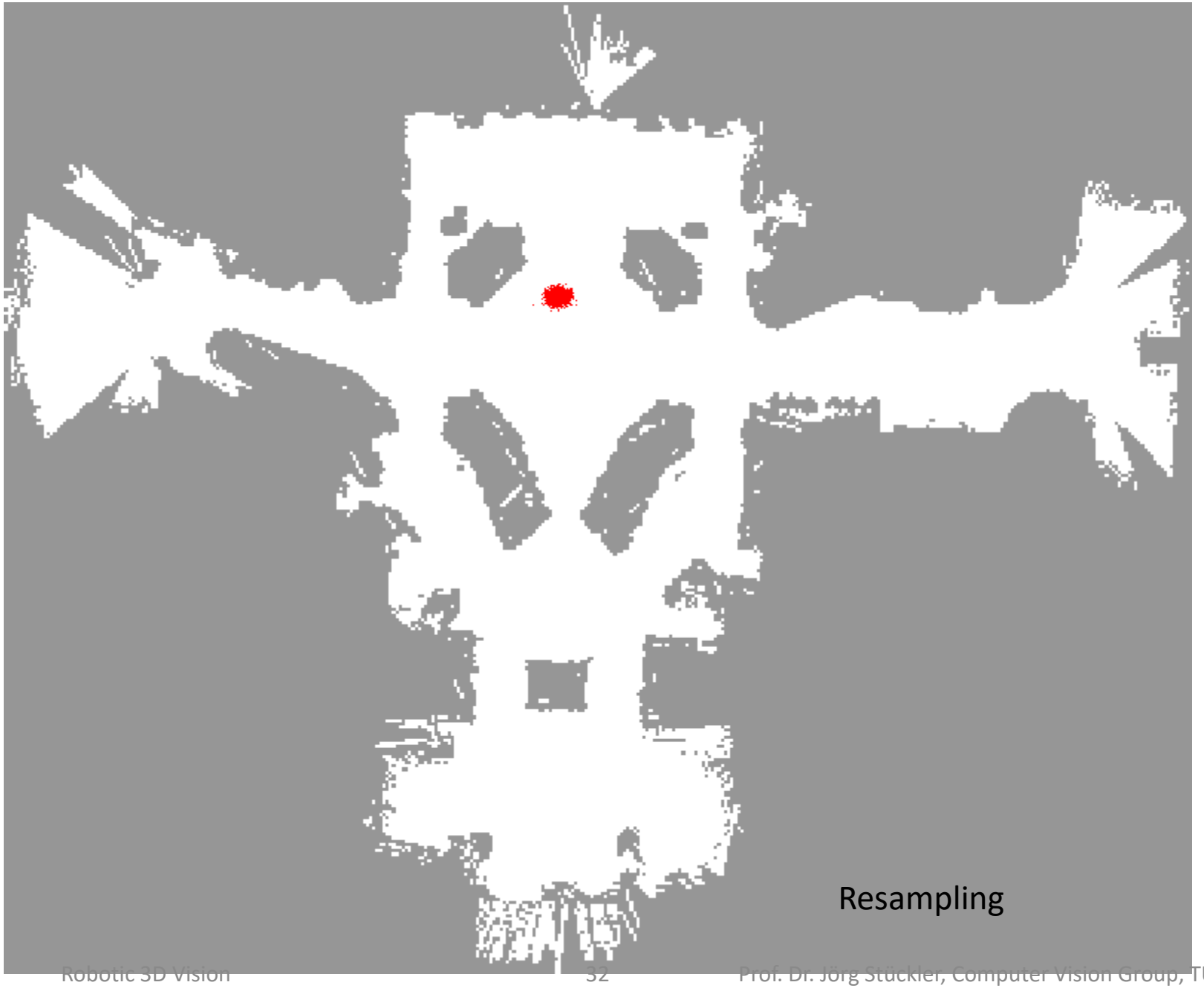
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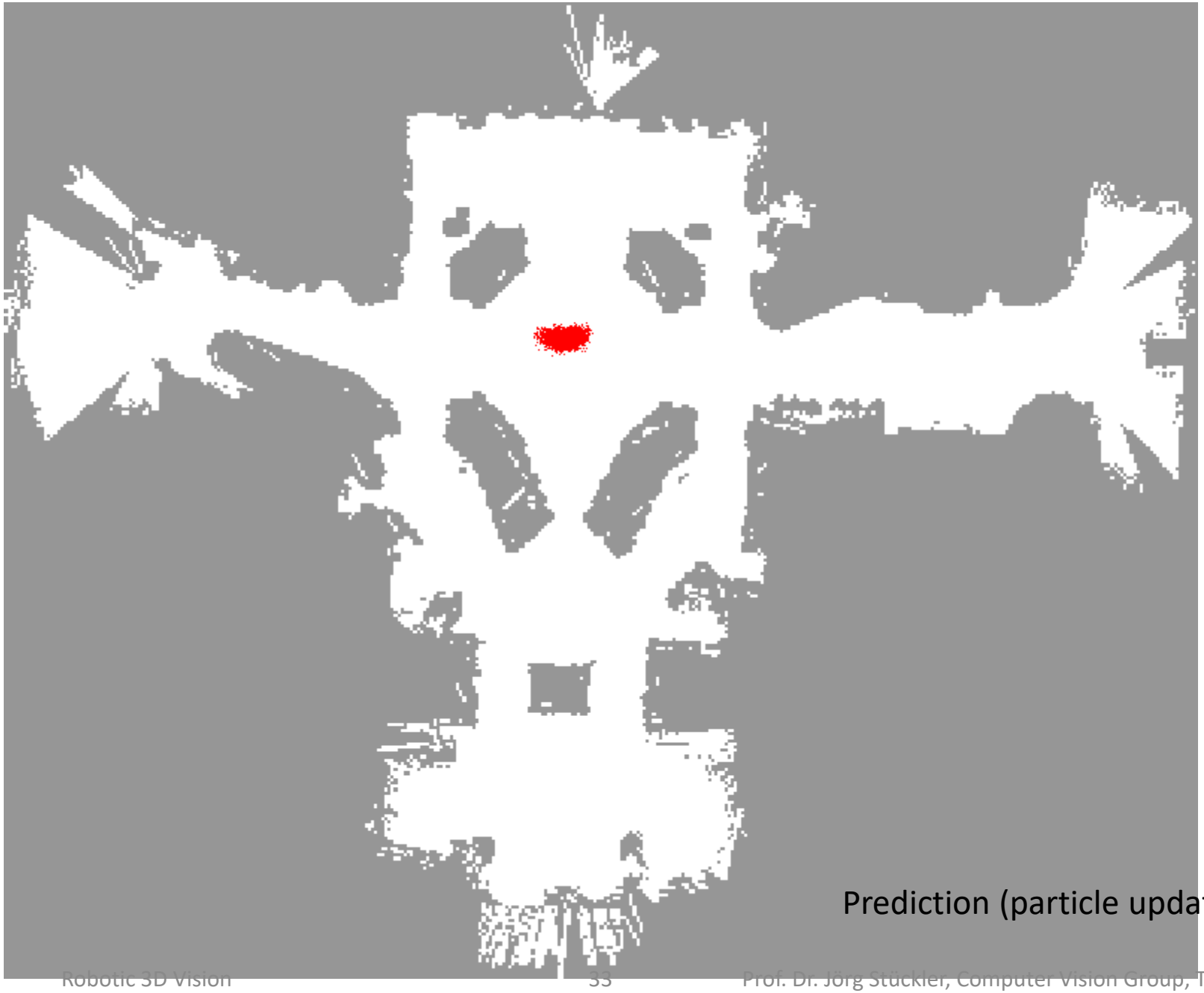

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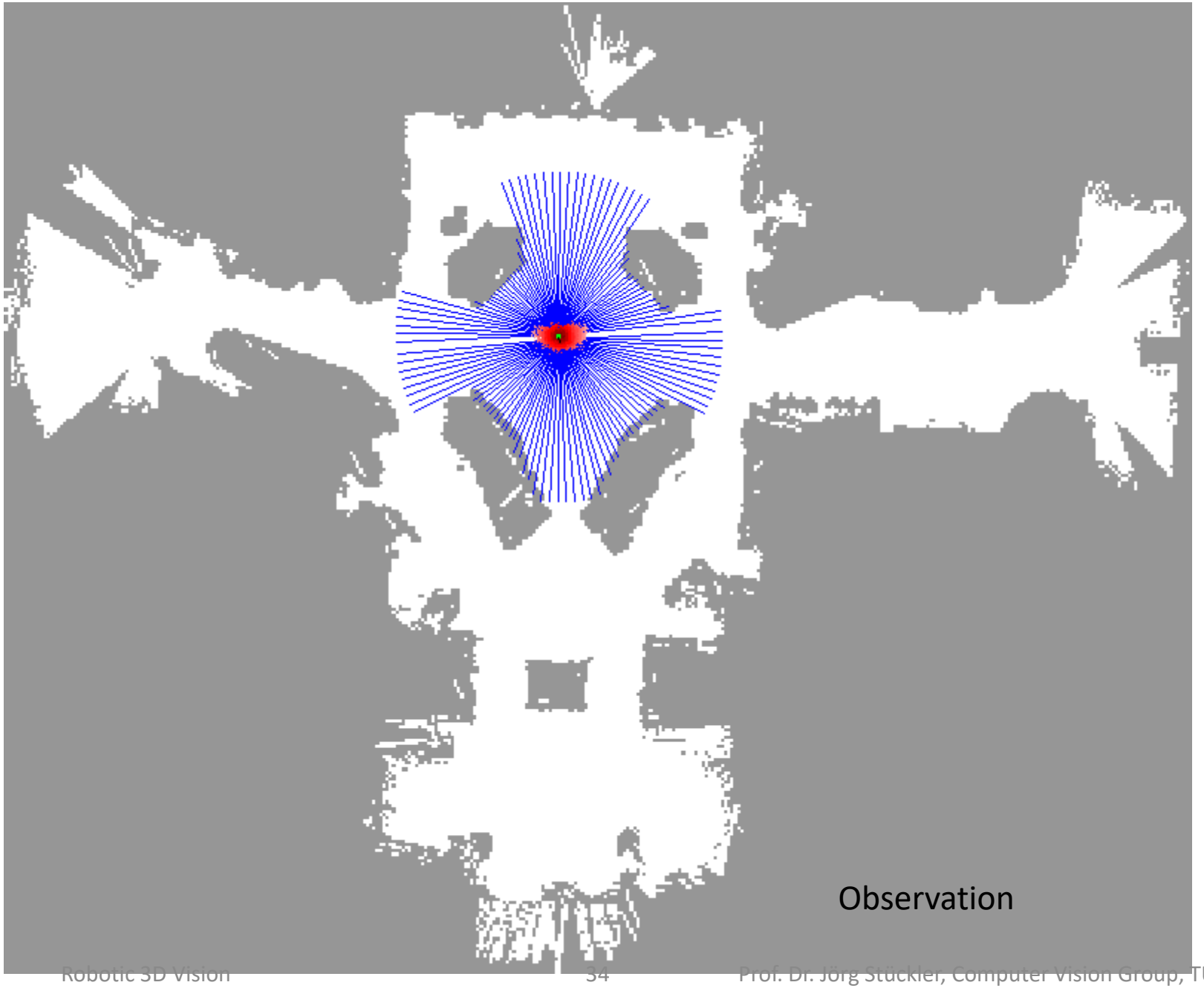
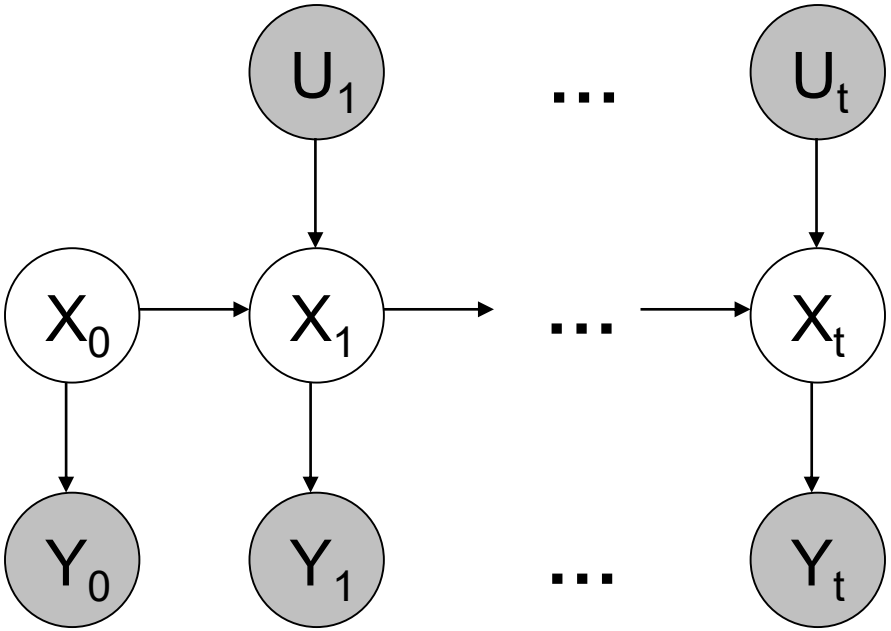

Observation

Short Recap of Directed Graphical Models

- What is the probabilistic semantics of a directed graphical model?
- Example of a directed graphical model:

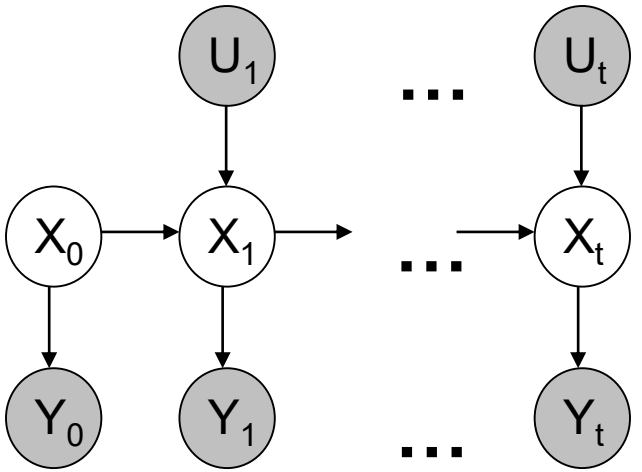


Directed Graphical Models

- Graph describes factorization of joint probability over random variables in terms of conditional probabilities
  - Nodes are random variables
  - Directed edges encode stochastic dependency relations

$$p(X_{0:t}, U_{1:t}, Y_{0:t}) = p(X_0)\left(\prod_{\tau=0}^{t} p(Y_\tau | X_\tau)\right)\left(\prod_{\tau=1}^{t} p(X_\tau | X_{\tau-1}, U_\tau) p(U_\tau)\right)$$

- Bayesian factorization into conditional probabilities of variables conditioned on their parents
- Graph needs to be acyclic
- Also called Bayesian network



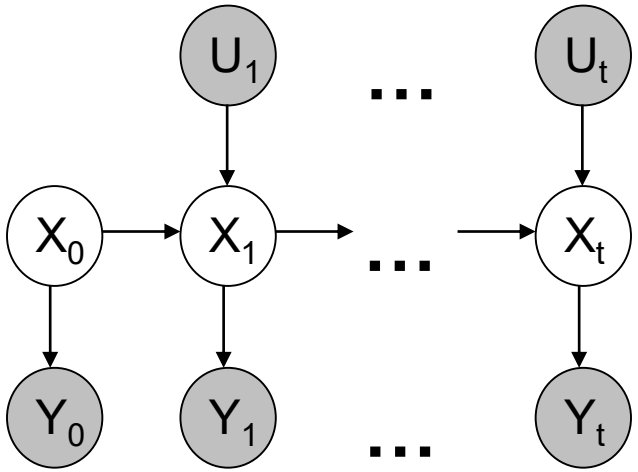
Semantics of Directed Graphical Models

- A directed graphical model expresses stochastic dependency relations between random variables
- Node sets X and Y are conditionally independent given set Z if Z d-separates X and Y in the graph:
  - Every undirected path between nodes in X and Y is blocked by nodes in Z
  - Path is blocked if there is a node W such that either
    - W has no converging arrows along path and W is in Z
    - W has converging arrows along path and neither W nor its descendants are in Z

$$X \perp\!\!\!\perp Y \mid Z$$

iff

$$p(X, Y \mid Z) = p(X \mid Z)\, p(Y \mid Z)$$



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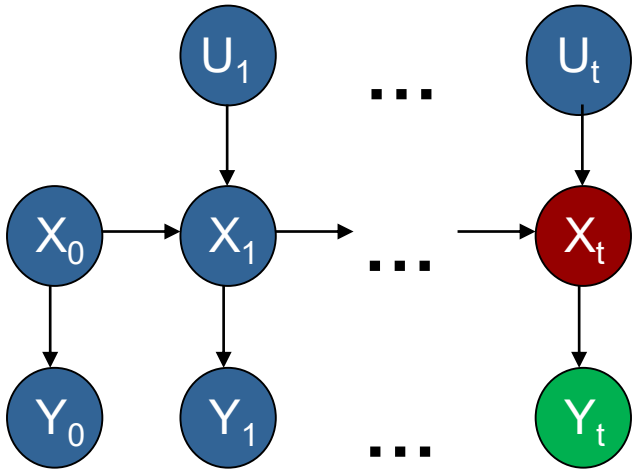

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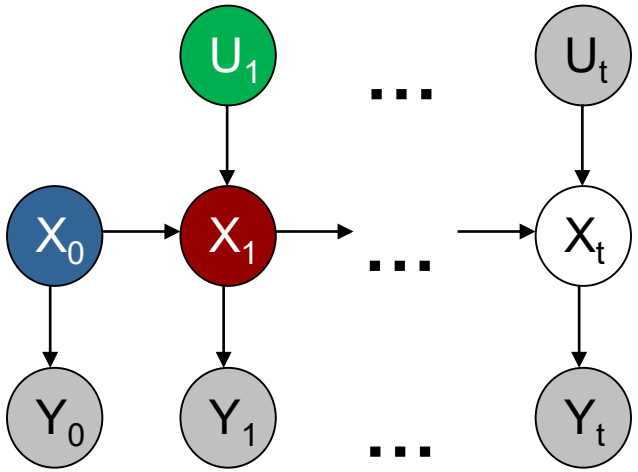

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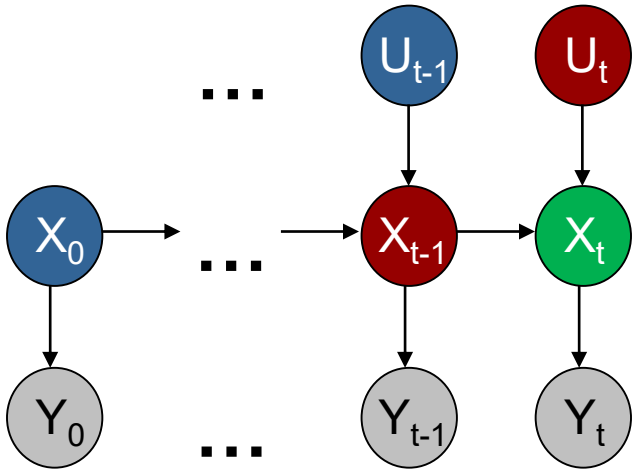


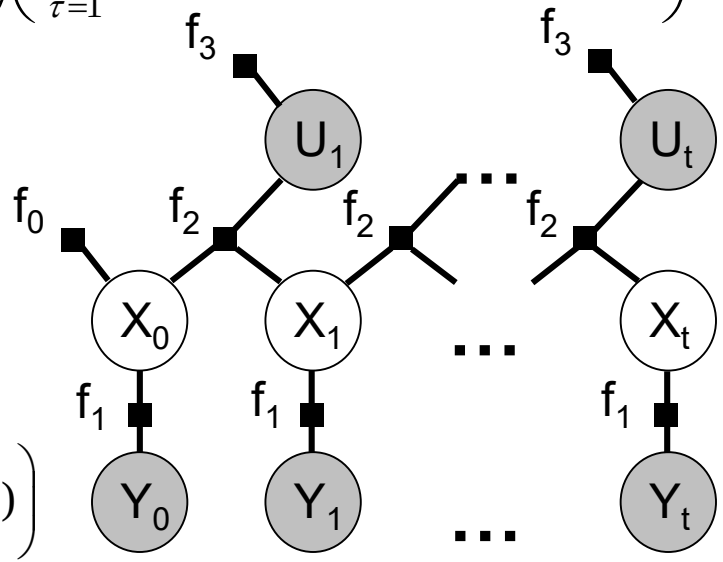
Factor Graphs

- Factor graphs also describe factorization of joint probability over random variables
  - Nodes are either random variables (discs) or factors (squares)
  - Undirected edges connect variables with factors

$$p(X_{0:t}, U_{1:t}, Y_{0:t}) = \frac{1}{Z} f_0(X_0) \left(\prod_{\tau=0}^{t} f_1(X_\tau, Y_\tau) \right) \left(\prod_{\tau=1}^{t} f_2(X_\tau, X_{\tau-1}, U_\tau) f_3(U_\tau) \right)$$

- Factors f_i are non-negative functions over variables
- Normalizer Z ensures that probability function integrates to 1

$$Z := \iiint_{X_{0:t}, Y_{0:t}, U_{1:t}} f_0(X_0) \left(\prod_{\tau=0}^{t} f_1(X_\tau, Y_\tau) \right) \left(\prod_{\tau=1}^{t} f_2(X_\tau, X_{\tau-1}, U_\tau) f_3(U_\tau) \right)$$



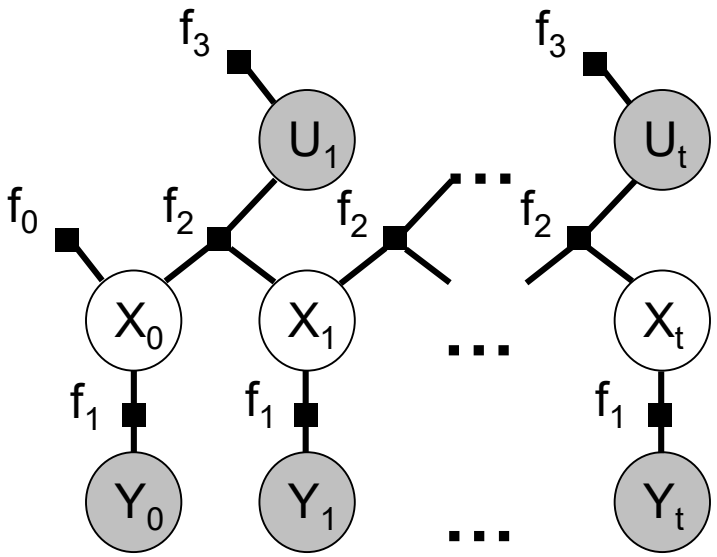
Semantics of Factor Graphs

- Factor graphs also express conditional independence relations
  - Two nodes are neighbors if they appear in a common factor
  - Path: sequence of neighboring nodes between two variables
  - Node set X is conditionally independent from set Y given set Z if all paths between nodes in X and Y are blocked by some node in Z
  - Given the set ne(X) of neighbors of X, the variable X is conditionally independent from the remaining variables

$$X \perp\!\!\!\perp Y \mid Z$$

iff

$$p(X, Y \mid Z) = p(X \mid Z)\, p(Y \mid Z)$$



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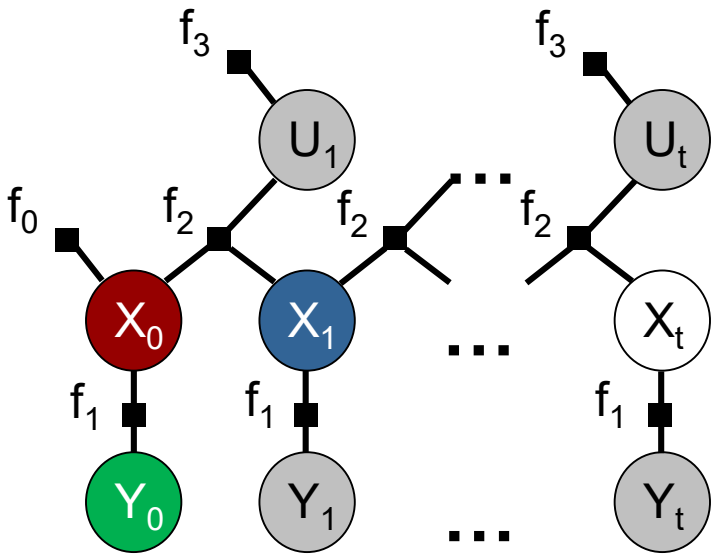
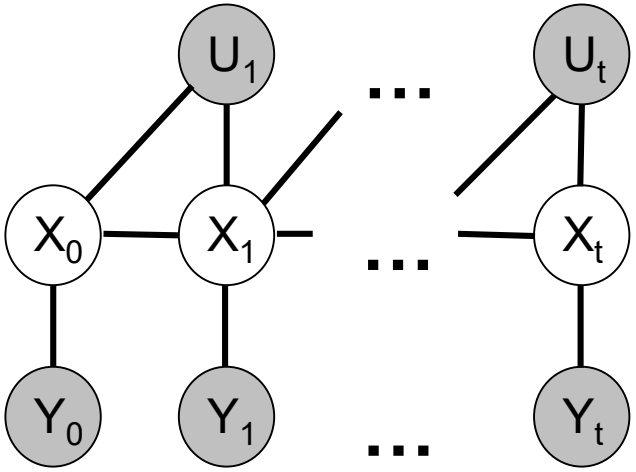
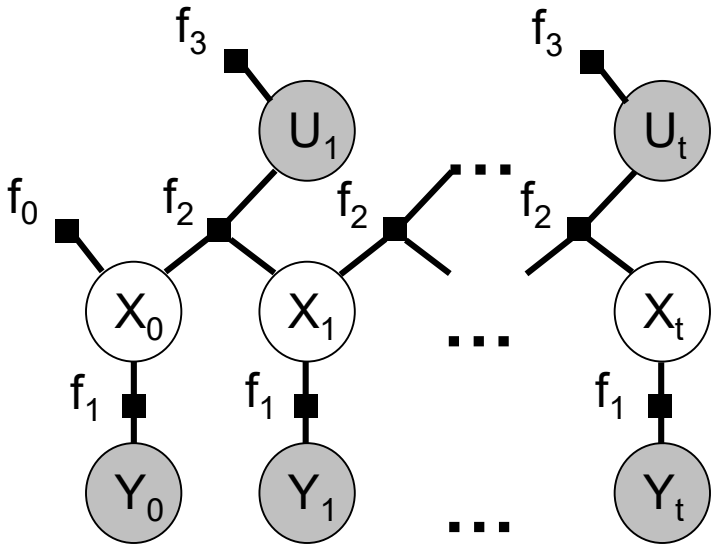


Factor Graphs and Undirected Graphical Models

- Another class of graphical models are undirected graphical models
  - Nodes are random variables (discs)
  - Undirected edges connect variables
- Undirected graphical models can be represented by factor graphs
  - Each factor node in a factor graph connects the cliques in an undirected graphical model





Factor Graphs vs. Directed Graphical Models

- In general, neither factor graphs nor directed graphical models can represent conditional independence relations of arbitrary probability distributions
- Both types of graphs have different expressiveness
- Examples on blackboard

Factor Graphs vs. Undirected Graphical Models

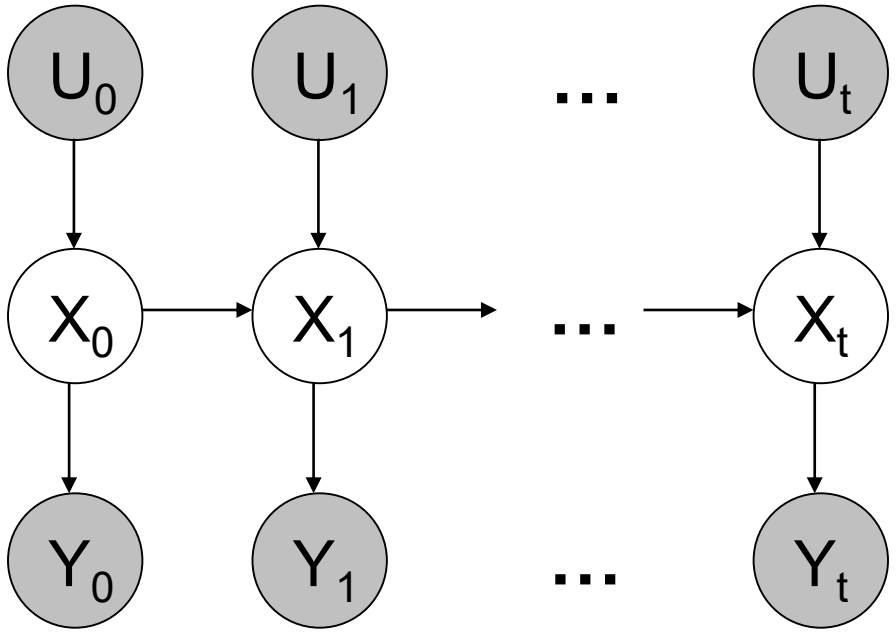
- Factor graphs and undirected graphical models have same expressiveness in terms of conditional independencies
- However, factor graphs can represent different granularities of factorizations for the same conditional dependencies
- Example on blackboard

Full State Posterior

- In filtering, we only consider most recent observation and action
- While Kalman filters are optimal for linear models and Gaussian noise, any approximation (EKF, PF, etc.) will introduce errors
- Can we perform inference for the full state $X_{0:t}$ given all observations $Y_{0:t}$ and actions $U_{0:t}$ so far?

$$p\left(X_{0:t} \middle| U_{0:t}, Y_{0:t}\right)$$

full state posterior



Full State Posterior Factorization

- The full state posterior factorizes into a product of observation likelihoods, state-transition likelihoods and the initial state distribution

$$
\begin{aligned}
p\left(X_{0:t} \middle| U_{1:t}, Y_{0:t}\right) &= \frac{p\left(Y_t \middle| X_{0:t}, U_{1:t}, Y_{0:t-1}\right) p\left(X_{0:t} \middle| U_{1:t}, Y_{0:t-1}\right)}{p\left(Y_t \middle| U_{1:t}, Y_{0:t}\right)} \\
&= \frac{p\left(Y_t \middle| X_t\right) p\left(X_t \middle| X_{0:t-1}, U_{1:t}, Y_{0:t-1}\right) p\left(X_{0:t-1} \middle| U_{1:t}, Y_{0:t-1}\right)}{p\left(Y_t \middle| U_{1:t}, Y_{0:t}\right)} \\
&= \eta_t \, p\left(Y_t \middle| X_t\right) p\left(X_t \middle| X_{t-1}, U_t\right) p\left(X_{0:t-1} \middle| U_{1:t-1}, Y_{0:t-1}\right) \\
&= p\left(X_0\right) \left(\prod_{\tau=0}^{t} \eta_\tau \, p\left(Y_\tau \middle| X_\tau\right) \right) \left(\prod_{\tau=1}^{t} p\left(X_\tau \middle| X_{\tau-1}, U_\tau\right) \right)
\end{aligned}
$$

Full State Posterior – Non-Linear Gaussian Case

$$p\left(X_{0:t} \middle| U_{1:t}, Y_{0:t}\right) = p\left(X_0\right)\left(\prod_{\tau=0}^{t} \eta_\tau \, p\left(Y_\tau \middle| X_\tau\right)\right)\left(\prod_{\tau=1}^{t} p\left(X_\tau \middle| X_{\tau-1}, U_\tau\right)\right)$$

- Non-linear state-transition model with Gaussian noise:

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{d_t})$$

$$p(\mathbf{x}_t \mid \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t; g(\mathbf{x}_{t-1}, \mathbf{u}_t), \boldsymbol{\Sigma}_{d_t})$$

- Non-linear observation model with Gaussian noise:

$$\mathbf{y}_t = h(\mathbf{x}_t) + \boldsymbol{\delta}_t$$

$$\boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{m_t})$$

$$p(\mathbf{y}_t \mid \mathbf{x}_t) = \mathcal{N}(\mathbf{y}_t; h(\mathbf{x}_t), \boldsymbol{\Sigma}_{m_t})$$

- Gaussian initial state estimate:

$$\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \qquad p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$$

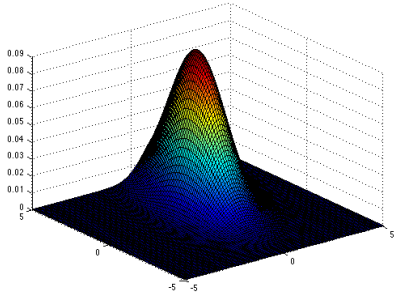
Full State Posterior – Non-Linear Gaussian Case

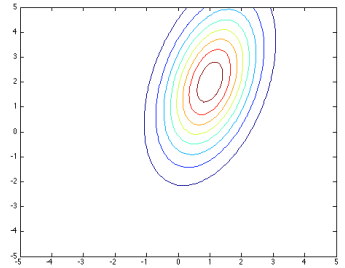
- We obtain the following factorization into normal distributions

$$p(\mathbf{x}_{0:t} \mid \mathbf{y}_{0:t}, \mathbf{u}_{1:t}) =$$
$$\mathcal{N}(\mathbf{x}_0; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \prod_{\tau=0}^{t} \eta_\tau \mathcal{N}(\mathbf{y}_\tau; h(\mathbf{x}_\tau), \boldsymbol{\Sigma}_{m_\tau}) \prod_{\tau=1}^{t} \mathcal{N}(\mathbf{x}_\tau; g(\mathbf{x}_{\tau-1}, \mathbf{u}_\tau), \boldsymbol{\Sigma}_{d_\tau})$$

- Recap: multivariate normal distribution

$$\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \left((2\pi)^d \det(\boldsymbol{\Sigma})\right)^{-1} \exp\left(-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$
$$d = \dim(\mathbf{x})$$





- How can we find the maximum a-posteriori estimate for $\mathbf{x}_{0:t}$?

Negative Log-Posterior

- Since the logarithm is a monotonic increasing function, we can minimize the negative log-posterior probability instead

$$E(\mathbf{x}_{0:t}) = \text{const.} + \tfrac{1}{2}(\mathbf{x}_0 - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}_0^{-1}(\mathbf{x}_0 - \boldsymbol{\mu}_0)$$
$$+ \tfrac{1}{2}\sum_{\tau=0}^{t}(\mathbf{y}_\tau - h(\mathbf{x}_\tau))^\top \boldsymbol{\Sigma}_{m_\tau}^{-1}(\mathbf{y}_\tau - h(\mathbf{x}_\tau))$$
$$+ \tfrac{1}{2}\sum_{\tau=1}^{t}(\mathbf{x}_\tau - g(\mathbf{x}_{\tau-1}, \mathbf{u}_\tau))^\top \boldsymbol{\Sigma}_{d_\tau}^{-1}(\mathbf{x}_\tau - g(\mathbf{x}_{\tau-1}, \mathbf{u}_\tau))$$

  - The exponential functions vanish
  - The normalization factors are independent of the state variables and can be subsumed in a constant
  - Constants do not contribute to the minimization problem
  - Quadratic function in non-linear residuals on $\mathbf{x}_{0:t}$
- Non-linear least squares problem!

Non-Linear Least Squares

- We can rewrite the negative log-posterior as a non-linear least squares problem:

$$\arg\min_{\mathbf{x}} E(\mathbf{x}) = \frac{1}{2}\mathbf{r}(\mathbf{x})^\top \mathbf{W}\mathbf{r}(\mathbf{x})$$

  - Stack residuals in residual vector $\mathbf{r}(\mathbf{x})$
  - Inverse covariances in block-diagonal weight matrix $\mathbf{W}$

- Optimization approaches:
  - Gradient descent
  - Gauss-Newton
  - Levenberg-Marquardt
  - etc.

Gradient Descent

- Idea 1: Perform gradient descent to minimize E(x)

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \eta \nabla E(\mathbf{x}_k)$$

$$\nabla E(\mathbf{x}) = \nabla \mathbf{r}(\mathbf{x})^\top \mathbf{W} \mathbf{r}(\mathbf{x})$$

- Pros:
  - Stable convergence for sufficiently small step size
- Cons:
  - Slow convergence (linear convergence rate)
  - Solution quality depends on initial guess

Gauss-Newton Method

- Idea 2: Approximate Newton's method to minimize E(x)
  - Approximate E(x) through linearization of residuals

$$\widetilde{E}(\mathbf{x}) = \frac{1}{2}\widetilde{\mathbf{r}}(\mathbf{x})^\top \mathbf{W}\widetilde{\mathbf{r}}(\mathbf{x})$$

$$= \frac{1}{2}\left(\mathbf{r}(\mathbf{x}_k) + \mathbf{J}_k\left(\mathbf{x}-\mathbf{x}_k\right)\right)^\top \mathbf{W}\left(\mathbf{r}(\mathbf{x}_k) + \mathbf{J}_k\left(\mathbf{x}-\mathbf{x}_k\right)\right) \qquad \mathbf{J}_k := \nabla_{\mathbf{x}}\mathbf{r}(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}_k}$$

$$= \frac{1}{2}\mathbf{r}(\mathbf{x}_k)^\top \mathbf{W}\mathbf{r}(\mathbf{x}_k) + \underbrace{\mathbf{r}(\mathbf{x}_k)^\top \mathbf{W}\mathbf{J}_k}_{=:\mathbf{b}_k^\top}\left(\mathbf{x}-\mathbf{x}_k\right) + \frac{1}{2}\left(\mathbf{x}-\mathbf{x}_k\right)^\top \underbrace{\mathbf{J}_k^\top \mathbf{W}\mathbf{J}_k}_{=:\mathbf{H}_k}\left(\mathbf{x}-\mathbf{x}_k\right)$$

  - Find root of $\nabla_{\mathbf{x}}\widetilde{E}(\mathbf{x}) = \mathbf{b}_k^\top + \left(\mathbf{x}-\mathbf{x}_k\right)^\top \mathbf{H}_k$ using Newton's method, i.e.

$$\nabla_{\mathbf{x}}\widetilde{E}(\mathbf{x}) = \mathbf{0} \text{ iff } \mathbf{x} = \mathbf{x}_k - \mathbf{H}_k^{-1}\mathbf{b}_k$$

- Pros:
  - Faster convergence (approx. quadratic convergence rate)
- Cons:
  - Divergence if too far from local optimum (H not positive definite)
  - Solution quality depends on initial guess

Levenberg-Marquardt Method

- Idea 3: Gradually switch between gradient descent and Gauss-Newton
  - Augment Hessian approximation of Gauss-Newton (damping)

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \left(\mathbf{H}_k + \lambda \mathbf{I}\right)^{-1} \mathbf{b}_k$$

  - Adaptive weighting: $\mathbf{x}_{k+1} = \mathbf{x}_k - \left(\mathbf{H}_k + \lambda \operatorname{diag}(\mathbf{H}_k)\right)^{-1} \mathbf{b}_k$
  - Start with $\lambda = 0.1$
  - Accept step and decrease lambda $\lambda \leftarrow \lambda/2$ if error function decreases, otherwise discard step and increase lambda $\lambda \leftarrow 2\lambda$ (akin line search)

- Pros:
  - Fast convergence close to local optimum (quadratic convergence rate close to optimum)
  - More stable but slow convergence far from local optimum
- Cons:
  - Solution quality depends on initial guess

Iteratively Reweighted Least Squares

- Adaptive weights allow for approximately optimizing other norms on the residual function

$$E(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{N} w_i \left\| r_i(\mathbf{x}) \right\|_2^2$$

- Example: for any l_p norm, we need to set $w_{i,k} = \left| r_i(\mathbf{x}_k) \right|^{p-2}$

Why Filter?

- Runtime depends polynomially on time (inversion of the Hessian)
- Estimating the full state posterior can become prohibitively slow
- Typical current approaches bound the runtime by approximations:
  - Selection and optimization of keyframes to limit the optimization window size (subsampling)
  - Marginalization of old states to keep number of optimized frames constant
  - In fact, marginalization of all old states but the current one corresponds to Kalman Filtering in the Gaussian noise case
- We will see approximations to full posterior optimization later in more detail in concrete examples

Tools and Frameworks

- C++
  - ceres (http://ceres-solver.org/)
  - g2o (https://github.com/RainerKuemmerle/g2o)
- Matlab
  - Optimization toolbox (e.g. lsqnonlin)
- Python
  - lmfit
  - scipy.optimize.curve_fit

Lessons Learned Today

- Directed graphical models and factor graphs describe stochastic conditional independence relations of joint probability distributions
- Full state posterior factorizes into conditional observation and state-transition likelihoods per time step
- Negative log-posterior leads to non-linear least squares in the case of Gaussian noise
- Levenberg-Marquardt method for robust pseudo second-order optimization of non-linear least squares problems
- Runtime of full posterior optimization grows polynomially with time
- Approximations will be needed

Thanks for your attention!