



Chapter 3

Variational Calculus

Computer Vision I: Variational Methods

Winter 2017/18

Variational Methods

Image Denoising

Iterative Solvers

Infinite-Dimensional
Setting

The Gâteaux
Derivative

The Euler-Lagrange
Equation

Gradient Descent

Boundary Conditions

Diffusion as Gradient
Descent

Euler and Lagrange

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Variational methods are a specific class of optimization methods. The key idea is to define cost functionals over a **continuous solution space** and to compute optima by solving the corresponding **extremality principle**.

Variational methods allow to solve respective problems in a **mathematically transparent** manner. Instead of performing a heuristic sequence of processing steps one starts by defining what properties a solution should have. Once these are fixed, the appropriate algorithm can be derived “automatically”.

Variational methods are particularly suited for infinite-dimensional problems. They are among the **top performing methods** for:

- image denoising, deblurring, super-resolution
- image segmentation
- motion estimation
- dense 3D reconstruction
- tracking



Advantages of Optimization Methods

Optimization methods have many advantages over traditional multi-step approaches (such as the Canny edge detector):

- A mathematical analysis of the cost function allows statements regarding the **existence**, **uniqueness** and **stability** of solutions to a given problem.
- In **traditional multistep processes** the interplay of consecutive steps is often **complex** and **intransparent**. It is typically unclear how modifying or replacing one component affects the subsequent steps.
- Optimization methods are based on **transparent and explicitly formulated assumptions**, with no “hidden” assumptions.
- In general, optimization methods have **fewer parameters**. The meaning of each parameter is fairly obvious.
- Optimization methods are **easily combined** in a transparent manner (by adding respective cost functions).



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A Simple Example: Image Denoising

Let $f : \Omega \rightarrow \mathbb{R}$ be an input greylevel image corrupted by noise. The goal is to compute a **denoised version u of the image f** .

The desired function u should fulfill two criteria:

- The function u should be **similar to f** .
- The function u should be **spatially smooth**.

Both criteria can be combined in the following cost function (or energy):

$$E(u) = E_{data}(u, f) + \lambda E_{smoothness}(u),$$

where the first term measures the similarity of u and f and the second term measures the smoothness of u . A weighting or **regularization parameter $\lambda \geq 0$** specifies the relative importance of smoothness versus data fit.

Most variational approaches have the above form. They merely differ in how the similarity term (**data term**) and the smoothness term (**regularizer**) are defined.





Let us assume for now that f is discrete in one spatial variable, i.e. $f = \{f_1, f_2, \dots, f_n\}$ is simply a sequence of brightness values $f_i \in \mathbb{R}$. We seek an approximation $u = \{u_1, \dots, u_n\}$.

The data term which measures similarity of f and u can for example be written as:

$$E_{data}(u) = \frac{1}{2} \sum_{i=1}^n (f_i - u_i)^2,$$

which means that we measure the overall brightness difference as a **sum of squared differences (SSD)**.

The smoothness term can for example be written as:

$$E_{smooth}(u) = \frac{1}{2} \sum_{i=1}^{n-1} (u_i - u_{i+1})^2,$$

which means that we measure the sum of squared differences for all neighboring brightness values.



The total energy is thus:

$$E_{\lambda}(u) = \frac{1}{2} \sum_{i=1}^n (f_i - u_i)^2 + \frac{\lambda}{2} \sum_{i=1}^{n-1} (u_i - u_{i+1})^2,$$

Larger values of λ imply that the smoothness of the solution should play a bigger role.

A solution to the above denoising problem is a function \hat{u} which minimizes the above energy:

$$\hat{u} = \arg \min_u E_{\lambda}(u).$$

Variational methods determine functions which fulfill the **extremality principle**:

$$\frac{dE_{\lambda}(u)}{du} = 0, \quad \Leftrightarrow \quad \frac{\partial E_{\lambda}(u)}{\partial u_i} = 0 \quad \forall i \in [1, n]$$

Discrete Denoising in 1D

The extremality condition for each pixel is therefore:

$$\frac{\partial E_\lambda(u)}{\partial u_1} = (u_1 - f_1) + \lambda(u_1 - u_2) = 0,$$

$$\frac{\partial E_\lambda(u)}{\partial u_i} = (u_i - f_i) + \lambda(2u_i - u_{i-1} - u_{i+1}) = 0, \quad \forall i \in [2, n-1],$$

$$\frac{\partial E_\lambda(u)}{\partial u_n} = (u_n - f_n) + \lambda(u_n - u_{n-1}) = 0.$$

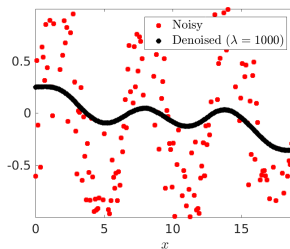
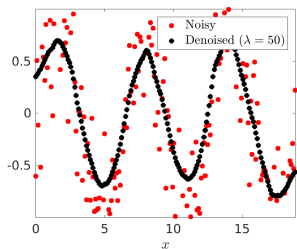
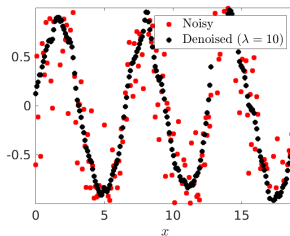
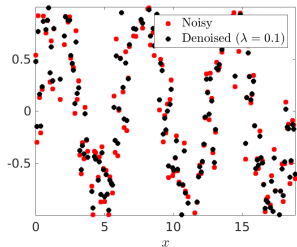
These conditions form a **system of linear equations**:

$$M_\lambda u = \begin{pmatrix} 1+\lambda & -\lambda & & & \\ -\lambda & 1+2\lambda & -\lambda & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda & 1+2\lambda & -\lambda \\ & & & -\lambda & 1+\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}$$



Discrete Denoising in 1D

Solution of $M_\lambda u = f$ for different λ values:

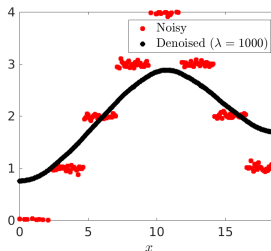
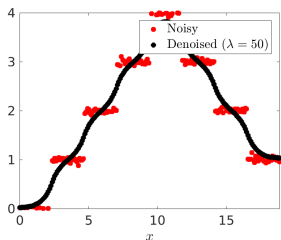
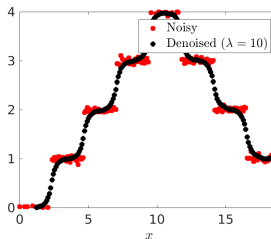
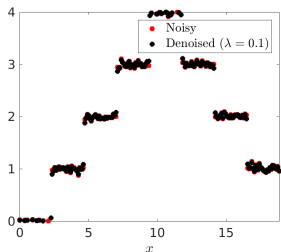


→ Tuning the regularization weight may be tedious



Discrete Denoising in 1D

Solution of $M_\lambda u = f$ for different λ values:



→ Quadratic regularization is not always a good choice (see next chapter)



- The matrix M_λ is tridiagonal.
- The inverse matrix M_λ^{-1} exists. It is dense and has only non-negative entries.
- The above system of equations can be solved in linear time using Gaussian elimination (Thomas' algorithm).
- The minimizer is **unique** because $E(u)$ is strictly convex.
- Reminder:
 - A **set** is called **convex**, if for any pair of points in the set, the connecting line is also contained in the set.
 - A **function** $E(u)$ is called **convex** if the function values are below the connecting line:

$$E((1-\alpha)u_1 + \alpha u_2) \leq (1-\alpha)E(u_1) + \alpha E(u_2) \quad \forall u_1, u_2 \quad \forall \alpha \in (0, 1).$$

- Remark: A function $E(u)$ is convex if and only if its epigraph is convex.

Proof of convexity

The function $h(s) = s^2$ is convex:

$$h((1-\alpha)s_1 + \alpha s_2) < (1-\alpha)h(s_1) + \alpha h(s_2), \quad \forall s_1, s_2, \forall \alpha \in (0, 1).$$

For any $u = (u_1, \dots, u_n)$ and $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)$ and for any $\alpha \in (0, 1)$ we therefore have:

$$\begin{aligned} E_\lambda((1-\alpha)u + \alpha\tilde{u}) &= \frac{1}{2} \sum_{i=1}^n \left(f_i - ((1-\alpha)u_i + \alpha\tilde{u}_i) \right)^2 \\ &\quad + \frac{\lambda}{2} \sum_{i=1}^{n-1} \left(((1-\alpha)u_i + \alpha\tilde{u}_i) - ((1-\alpha)u_{i+1} + \alpha\tilde{u}_{i+1}) \right)^2 \\ &= \frac{1}{2} \sum_{i=1}^n \left((1-\alpha)(f_i - u_i) + \alpha(f_i - \tilde{u}_i) \right)^2 \\ &\quad + \frac{\lambda}{2} \sum_{i=1}^{n-1} \left(((1-\alpha)(u_i - u_{i+1}) + \alpha(\tilde{u}_i - \tilde{u}_{i+1})) \right)^2 \\ &< (1-\alpha)E_\lambda(u) + \alpha E_\lambda(\tilde{u}). \end{aligned}$$



Discrete Denoising (d -dim.)

Index all pixels of the d -dim volume with index $i \in [1, \dots, N]$, where $N = n_1 \cdot n_2 \cdots n_d$.

Variational denoising of an image f :

$$E_\lambda(u) = \frac{1}{2} \sum_{i=1}^N (f_i - u_i)^2 + \frac{\lambda}{2} \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}(i) \\ j > i}} (u_i - u_j)^2,$$

where $\mathcal{N}(i)$ denotes the **neighborhood of pixel i** (typically, we consider 4-connexity in 2D).

Again E is strictly convex. The condition for (global) optimality is:

$$\frac{dE_\lambda(u)}{du_i} = (u_i - f_i) + \lambda \sum_{j \in \mathcal{N}(i)} (u_i - u_j) = 0 \quad \forall i$$

In the higher-dimensional case, this gives rise to a large-scale linear programming problem of the form $M_\lambda u = f$.



Discrete Denoising in 2D

Solution of $M_\lambda u = f$ for different λ values:

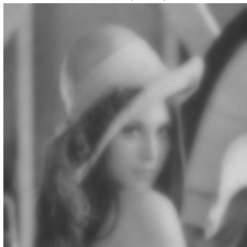
Denoised ($\lambda = 0.1$)



Denoised ($\lambda = 10$)



Denoised ($\lambda = 50$)



Denoised ($\lambda = 1000$)





For dimensions ($d = 2, 3, \dots$), the linear equation system

$$M_\lambda u = f$$

is quite large.

Most entries of the matrix M_λ are 0 (**sparse matrix**). Yet, its inverse is typically difficult to compute or even to store in memory.

There exist numerous standard solvers for large linear systems. The best known ones are the **Jacobi method** and the **Gauss-Seidel method**.

In the following, we will discuss the Jacobi solver, the Gauss-Seidel solver and various extensions.

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The Jacobi Method

The Jacobi method converges if the matrix $M \equiv M_\lambda$ is **strictly diagonally dominant**, i.e. if for any row of the matrix the absolute value of the diagonal element is larger than the sum of absolute values of the off-diagonal elements:

$$|m_{ii}| > \sum_{j \neq i} |m_{ij}| \quad \forall i$$

Decompose the matrix $M = D + A$ into its **diagonal part** D and the **offdiagonal part** A . Then:

$$Mu = (D + A)u = f \quad \Leftrightarrow \quad Du = f - Au$$

Initialize with an arbitrary function u^0 and iterate:

$$u^{(k+1)} = D^{-1}(f - Au^{(k)}), \quad k = 0, 1, 2, 3, \dots$$

Low computational and memory cost, easily parallelizable.

We can use the residuum $r^{(k)} := Mu^{(k)} - f$ as a **termination criterion**: $\frac{|r^{(k)}|}{|r^{(0)}|} < \epsilon$.





- Separate diagonal part D :

$$u_i + \lambda |\mathcal{N}(i)| u_i = f_i + \lambda \sum_{j \in \mathcal{N}(i)} u_j \quad \forall i,$$

where $|\mathcal{N}(i)|$ is the number of neighbors of pixel i .

- Iteration:

$$u_i^{(k+1)} = \frac{f_i + \lambda \sum_{j \in \mathcal{N}(i)} u_j^{(k)}}{1 + \lambda |\mathcal{N}(i)|}$$

- Appropriate initialization:

$$u^{(0)} = f$$

- **Intuitive Interpretation:** For each pixel i , the above iteration induces a weighted averaging of brightness values in its neighborhood $\mathcal{N}(i)$. The weights sum to 1.

Alternative Methods

Besides the Jacobi method, there are other methods for solving linear equation systems of the form $Mu = f$. In particular:

- **Gauss-Seidel method:** Updates pixel values sequentially always using the most recent values:

$$u_i^{(k+1)} = \frac{1}{m_{ii}} \left(f_i - \sum_{j < i} m_{ij} u_j^{(k+1)} - \sum_{j > i} m_{ij} u_j^{(k)} \right)$$

Faster than the Jacobi method. But: not parallelizable, solution depends on the order of updates.

- **SOR method (Successive Over-Relaxation):** Extrapolate the Gauss-Seidel updates linearly for faster convergence:

$$u_i^{(k+1)} = \omega \bar{u}_i^{(k+1)} + (1 - \omega) u_i^{(k)}$$

where $\bar{u}_i^{(k+1)}$ is the current solution of the Gauss-Seidel method, and $\omega \in [1, 2)$ the extrapolation factor ($\omega = 1$: Gauss-Seidel).





There exists a multitude of numerical strategies to accelerate algorithms. Two common examples are:

- **Preconditioned Conjugate Gradients:** Modification of the gradient descent or steepest descent which minimizes along orthogonal directions. Preconditioning can additionally improve the convergence rate.
- **Multigrid Methods:** Solve the equation system starting from a coarse-grid representation and use solution as initialization on respective finer grids.

Which combination of methods leads to the fastest solution depends on the type of optimization problem / cost function and the available hardware.

For example, the Gauss-Seidel and SOR methods are typically faster than the Jacobi method, but they are not directly parallelizable.

Alternative Methods

Speed of convergence for different solvers:

Variational Calculus

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A spatially continuous variational approach to image denoising and restoration looks as follows:

Given an image $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ find a smooth approximation $u : \Omega \rightarrow \mathbb{R}$ of this image. This can be determined by minimizing a cost function of the form:

$$E(u) = \frac{1}{2} \int (u(x) - f(x))^2 dx + \frac{\lambda}{2} \int |\nabla u(x)|^2 dx$$

This is the spatially continuous analogue of the previously discussed discrete formulation.

Such a mapping which assigns a real number $E(u)$ to a function $u(x)$ is also referred to as a **functional**.

How does one minimize the above functional with respect to the function u ?



A **functional** is a mapping E which assigns to each element of a vector-space (to each function u) an element from the underlying field (a number).

Let

$$E(u) = \int \mathcal{L}(u, u') dx$$

be a functional, where $u' = \frac{du}{dx}$ is the derivative of the function u . (In physics \mathcal{L} is called the **Lagrange density**).

Example: $\mathcal{L}(u, u') = \frac{1}{2}(u(x) - f(x))^2 + \frac{\lambda}{2}|u'(x)|^2$.

Just as with real-valued functions defined on \mathbb{R}^n the necessary condition for extremality of the functional E states that the **derivative with respect to u must be 0**.

Yet how does one define and compute the derivative of a functional $E(u)$ with respect to the function u ?

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The Gâteaux Derivative

There are several ways to introduce functional derivatives. The following definition goes back to works of the French mathematician **R. Gâteaux** († 1914) which were published posthumously in 1919: [http:](http://archive.numdam.org/ARCHIVE/BSMF/BSMF_1919__47_/BSMF_1919__47__47_1/BSMF_1919__47__47_1.pdf)

[//archive.numdam.org/ARCHIVE/BSMF/BSMF_1919__47_/BSMF_1919__47__47_1/BSMF_1919__47__47_1.pdf](http://archive.numdam.org/ARCHIVE/BSMF/BSMF_1919__47_/BSMF_1919__47__47_1/BSMF_1919__47__47_1.pdf)

The Gâteaux derivative extends the concept of directional derivative to infinite-dimensional spaces.

The derivative of the functional $E(u)$ in direction $h(x)$ is defined as:

$$\left. \frac{dE(u)}{du} \right|_h = \lim_{\epsilon \rightarrow 0} \frac{E(u + \epsilon h) - E(u)}{\epsilon}$$

As in finite dimensions, this **directional derivative** can be interpreted as the **projection of the functional gradient on the respective direction**. We can therefore write:

$$\left. \frac{dE(u)}{du} \right|_h = \left\langle \frac{dE(u)}{du}, h \right\rangle = \int \frac{dE(u)}{du}(x) h(x) dx$$



The Gâteaux Derivative

For functionals of the **canonical form**: $E(u) = \int \mathcal{L}(u, u') dx$ the Gâteaux derivative is given by

$$\begin{aligned} \left. \frac{dE(u)}{du} \right|_h &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (E(u + \epsilon h) - E(u)) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int (\mathcal{L}(u + \epsilon h, u' + \epsilon h') - \mathcal{L}(u, u')) dx \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int \left(\left(\mathcal{L}(u, u') + \frac{\partial \mathcal{L}}{\partial u} \epsilon h + \frac{\partial \mathcal{L}}{\partial u'} \epsilon h' + o(\epsilon^2) \right) - \mathcal{L}(u, u') \right) dx \\ &= \int \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial u'} h' \right) dx \\ &= \int \left(\frac{\partial \mathcal{L}}{\partial u} h - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} h \right) dx \quad (\text{partial int., } h = 0 \text{ on boundary}) \\ &= \int \underbrace{\left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \right)}_{\frac{dE}{du}} h(x) dx. \end{aligned}$$



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Euler-Lagrange Equation

Thus the derivative of the functional $E(u)$ in direction h is:

$$\left. \frac{dE(u)}{du} \right|_h = \int \underbrace{\left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \right)}_{\frac{dE}{du}} h(x) dx.$$

As a necessary condition for minimality of the functional $E(u)$ the **variation of E in any direction $h(x)$ must vanish**. Therefore at the extremum we have:

$$\boxed{\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = 0}$$

This condition is called the **Euler-Lagrange equation**.

Example: For $\mathcal{L}(u, u') = \frac{1}{2}(u(x) - f(x))^2 + \frac{\lambda}{2}|u'(x)|^2$, we get:

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = (u(x) - f(x)) - \frac{d}{dx}(\lambda u'(x)) = u - f - \lambda u'' = 0$$





The Euler-Lagrange equation is a differential equation which forms the **necessary condition for minimality**.

The central idea of variational methods is to compute solutions to the respective Euler-Lagrange equation.

This can be done in several ways. For example, one can discretize the function u on a set of points $\{x_1, \dots, x_n\}$ and subsequently try to solve for the values $u(x_i)$. For quadratic cost functions, the arising set of linear equations can be solved using the discussed iterative algorithms (Jacobi, Gauss-Seidel,...). In general, however, the Euler-Lagrange equation will not be linear.

For general (non-quadratic) energies, one can start with an initial guess $u_0(x)$ of the solution and iteratively improve the solution. Such methods are called **descent methods**.

How can one iteratively improve a given solution?

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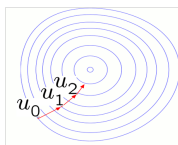
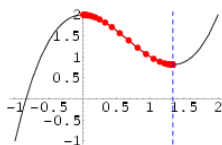
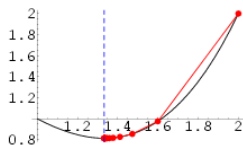
Gradient Descent

Gradient descent or steepest descent is a particular descent method where in each iteration one chooses the direction in which the energy decreases most. The **direction of steepest descent** is given by the **negative energy gradient**.

To minimize a real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the gradient descent for $f(u)$ is defined by the differential equation:

$$\begin{cases} u(0) = u_0 \\ \frac{du}{dt} = -\frac{df}{du}(u) \end{cases}$$

Discretization: $u_{t+1} = u_t - \epsilon \frac{df}{du}(u_t), \quad t = 0, 1, 2, \dots$



Gradient Descent

For minimizing functionals $E(u)$, the gradient descent is done analogously.

For the functional $E(u) = \int \mathcal{L}(u, u') dx$, the gradient is given by:

$$\frac{dE}{du} = \frac{d\mathcal{L}}{du} - \frac{d}{dx} \frac{d\mathcal{L}}{du'}.$$

Therefore the gradient descent is given by:

$$\begin{cases} u(x, 0) = u_0(x) \\ \frac{\partial u(x, t)}{\partial t} = -\frac{dE}{du} = -\frac{d\mathcal{L}}{du} + \frac{\partial}{\partial x} \frac{d\mathcal{L}}{du'}. \end{cases}$$

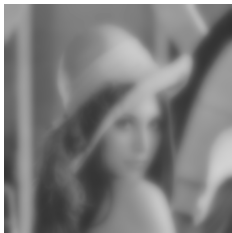
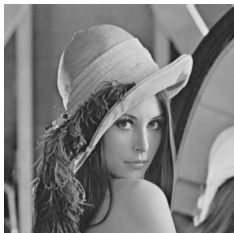
For $\mathcal{L}(u, u') = \frac{1}{2}(u - f)^2 + \frac{\lambda}{2}|u'|^2$, this means:

$$\frac{\partial u}{\partial t} = (f - u) + \lambda u'' = (f - u) + \lambda \Delta u.$$

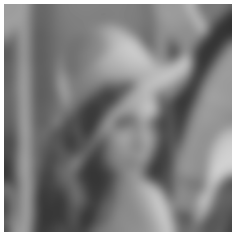
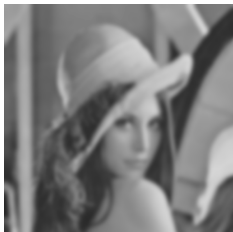
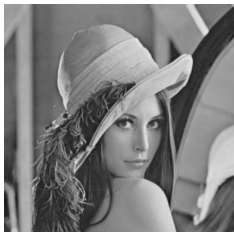
If the gradient descent converges, i.e. $\partial_t u = -\frac{dE}{du} = 0$, then we have found a solution to the Euler-Lagrange equation.



Diffusion with Data Term



$$E(u) = \int (f - u)^2 dx + \lambda \int |\nabla u|^2 dx \rightarrow \min.$$



$$E(u) = \lambda \int |\nabla u|^2 dx \rightarrow \min.$$

Author: D. Cremers



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Addendum: Boundary Conditions

When deriving the Euler-Lagrange equations we only considered perturbations $h(x)$ which are 0 on the boundary.

Without this assumption, Gâteaux's directional derivative is:

$$\begin{aligned}\frac{dE(u)}{du}\Big|_h &= \dots = \int_a^b \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial u'} h' \right) dx \\ &= \int_a^b \left(\frac{\partial \mathcal{L}}{\partial u} h - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} h \right) dx + \left(\frac{\partial \mathcal{L}}{\partial u'} h(x) \right)_a^b \\ &= \int_a^b \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \right) h(x) dx + \left(\frac{\partial \mathcal{L}}{\partial u'} h(x) \right)_a^b = 0\end{aligned}$$

$$\Rightarrow \begin{cases} 1) & \frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = 0 \\ 2) & \left(\frac{\partial \mathcal{L}}{\partial u'} h(x) \right)_a^b = 0 \end{cases}$$



Addendum: Boundary Conditions

Depending on the application one can distinguish two kinds of boundary conditions:

- **Dirichlet boundary conditions:** The function $u(x)$ is fixed on the boundary ($u_r(x)$), i.e. $h(x) = 0$ on the boundary. One only considers variations of $u(x)$ inside the domain:

$$\begin{cases} \frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = 0 \\ u(x) \Big|_{\text{boundary}} = u_r(x) \end{cases}$$

- **Neumann boundary conditions:** One additionally allows for variations of $u(x)$ on the boundary:

$$\begin{cases} \frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = 0 \\ \frac{\partial \mathcal{L}}{\partial u'} \Big|_{\text{boundary}} = 0 \end{cases}$$



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Many **diffusion equations** (albeit not all) can be derived as the **gradient descent** on a specific energy.

The energy:

$$E(u) = \int \mathcal{L}(u, \nabla u) dx = \frac{1}{2} \int g(x) |\nabla u(x)|^2 dx$$

leads to the gradient descent:

$$\frac{\partial u(x, t)}{\partial t} = -\frac{dE}{du} = -\frac{\partial \mathcal{L}}{\partial u} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla u} = \nabla \cdot (g(x) \nabla u)$$

This equation corresponds to an **inhomogeneous diffusion** with diffusivity $g(x)$.

In other words, the above inhomogeneous diffusion process is nothing but a steepest descent on the weighted smoothness energy.

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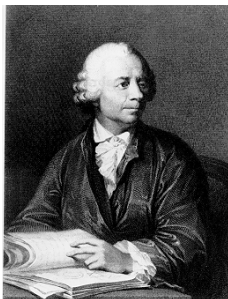
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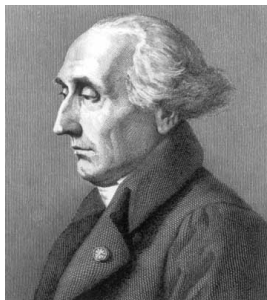
Euler and Lagrange



Leonhard Euler (1707 – 1783)

- Published 886 papers and books, most of them in the last 20 years of his life. Considered the greatest mathematician of the 18th century.
- Major contributions: Euler number e , Euler angles, Euler formula, Euler theorem, Euler equations (for fluid flows), Euler-Lagrange equations,...
- 13 children





Joseph-Louis Lagrange (1736 – 1813)

- born Giuseppe Lodovico Lagrangia (in Turin). self-taught.
- With 17 years: Professor for mathematics in Turin.
- Later in Berlin (1766-1787) and Paris (1787-1813).
- 1788: *La Mécanique Analytique*.
- 1800: *Leçons sur le calcul des fonctions*.



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