



Chapter 4

Variational Image Restoration

Computer Vision I: Variational Methods

Winter 2017/18

Inverse Problems and
Image Restoration

Image Denoising

Image Deblurring

Inverse Problems and
Bayesian Inference

Motion Blur and
Defocus Blur

Video Super
Resolution

Inpainting

Dr. Yvain QUÉAU
Chair for Computer Vision and Pattern Recognition
Departments of Informatics & Mathematics
Technical University of Munich



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Inverse Problems, Ill-Posedness and Regularization

In mathematics, the conversion of measurement data into information about the observed object or the observed physical system is referred to as an **inverse problem**.



Following **Hadamard (1902)**, a mathematical problem is called **well-posed** iff:

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- 1 A solution exists.
- 2 The solution is unique.
- 3 The solution's behavior changes continuously with the initial conditions.

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Inverse problems are often **ill-posed**. Since the measurement data is often not sufficient to uniquely characterize the observed object or system, one introduces **prior knowledge** to disambiguate which solutions are a priori more likely. In the context of variational methods this prior knowledge gives rise to the **regularity term**.

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Image Restoration: Denoising

Image restoration is a classical inverse problem: Given an observed image $f : \Omega \rightarrow \mathbb{R}$ and a (typically stochastic) model of an **image degradation process**, we want to restore the original image $u : \Omega \rightarrow \mathbb{R}$.

Image denoising is an example of image restoration where we assume that the true image u is corrupted by (additive) noise:

$$f = u + \eta, \quad \eta \sim \mathcal{N}(0, \sigma).$$

Rudin, Osher, Fatemi (1992) denoise f by minimizing a quadratic data term with **Total Variation (TV)** regularization:

$$\min_u \frac{1}{2} \int |u - f|^2 dx + \int |\nabla u| dx.$$

This gives rise to the **Euler-Lagrange equation**

$$u - f - \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0.$$

Other noise models and regularizers are conceivable.



Image Restoration: Denoising



original



noisy



denoised

(Goldlücke, Strelakovski, Cremers, SIAM J. Imaging Sci. '12)



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Image Restoration: Deblurring

A prototypical **blur model** is given by

$$f = A * u + \eta \quad \eta \sim \mathcal{N}(0, \sigma),$$

with a blur kernel A .

In a variational setting, this process can be inverted by minimizing the **TV deblurring functional**:

$$\min_u \frac{1}{2} \int |A * u - f|^2 dx + \int |\nabla u| dx.$$

For symmetric kernels A , the **Euler-Lagrange equation** is given by:

$$A * (A * u - f) - \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0,$$

and the **gradient descent equation**

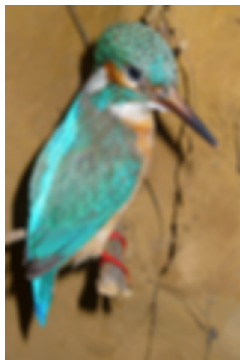
$$\frac{\partial u}{\partial t} = -A * (A * u - f) + \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right).$$



Image Restoration: Deblurring



Original



blurred and noisy



deblurred

(Goldluecke, Cremers, ICCV 2011)

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Inverse Problems and Bayesian Inference

The framework of **Bayesian inference** allows to systematically derive functionals for different image formation models.

Let u be the unknown true image and f the observed one, then we can write the joint probability for u and f as:

$$\mathcal{P}(u, f) = \mathcal{P}(u|f) \mathcal{P}(f) = \mathcal{P}(f|u) \mathcal{P}(u).$$

Rewriting this expression we obtain the **Bayesian formula** (Thomas Bayes 1887):

$$\mathcal{P}(u|f) = \frac{\mathcal{P}(f|u) \mathcal{P}(u)}{\mathcal{P}(f)}.$$

Maximum A posteriori (MAP) estimation aims at computing the most likely solution \hat{u} given f by maximizing the **posterior probability** $\mathcal{P}(u|f)$

$$\hat{u} = \arg \max_u \mathcal{P}(u|f) = \arg \max_u \mathcal{P}(f|u) \mathcal{P}(u).$$

$\mathcal{P}(f|u)$ is called the **likelihood** and $\mathcal{P}(u)$ the **prior**.



MAP Estimation in the Discrete Setting

Let us assume n independent pixels. For each the **measured intensity** f_i is given by the **true intensity** u_i plus **additive Gaussian noise**. This corresponds to the likelihood

$$\mathcal{P}(f_i|u_i) \propto \exp\left(-\frac{(u_i - f_i)^2}{2\sigma^2}\right).$$

Since all measurements are mutually independent, we obtain for the entire vector $f = (f_1, \dots, f_n)$ of pixel intensities:

$$\mathcal{P}(f|u) = \prod_{i=1}^n \mathcal{P}(f_i|u) = \prod_{i=1}^n \mathcal{P}(f_i|u_i) \propto \prod_{i=1}^n \exp\left(-\frac{(u_i - f_i)^2}{2\sigma^2}\right).$$

We now expand the prior:

$$\mathcal{P}(u) = \mathcal{P}(u_1 \dots u_n) = \mathcal{P}(u_1|u_2 \dots u_n)\mathcal{P}(u_2 \dots u_n) \propto \prod_{i=1}^{n-1} \mathcal{P}(u_i|u_{i+1}),$$

where we assumed a **Markov property**, namely that the probability of u_i is sufficiently characterized by its neighbor.



MAP Estimation in the Discrete Setting

Assuming a simple **smoothness prior**, we have:

$$\mathcal{P}(u) \propto \prod_{i=1}^{n-1} \mathcal{P}(u_i | u_{i+1}) \propto \prod_{i=1}^{n-1} \exp(-\lambda |u_i - u_{i+1}|).$$

With these assumptions, the **posterior distribution** is given by:

$$\mathcal{P}(u|f) \propto \prod_{i=1}^n \exp\left(-\frac{|f_i - u_i|^2}{2\sigma^2}\right) \prod_{i=1}^{n-1} \exp(-\lambda |u_i - u_{i+1}|)$$

Rather than maximizing this probability distribution, one can equivalently **minimize its negative logarithm** (because the logarithm is strictly monotonous).

It is given by the **energy**

$$E(u) = -\log \mathcal{P}(u|f) = \sum_{i=1}^n \frac{|f_i - u_i|^2}{2\sigma^2} + \lambda \sum_{i=1}^{n-1} |u_i - u_{i+1}| + \text{const.}$$



MAP Estimation in the Continuous Setting

Similarly one can define **Bayesian MAP optimization in the continuous setting**, where the likelihood is given by:

$$\mathcal{P}(f|u) \propto \exp\left(-\int \frac{|f(x) - u(x)|^2}{2\sigma^2} dx\right),$$

and the prior is given by

$$\mathcal{P}(u) \propto \exp\left(-\lambda \int |\nabla u(x)| dx\right).$$

Thus the data term in variational methods corresponds to the likelihood, whereas the regularizer corresponds to the prior:

$$E(u) = -\log \mathcal{P}(u|f) = \int \frac{|f(x) - u(x)|^2}{2\sigma^2} dx + \lambda \int |\nabla u(x)| dx + \text{const.}$$

A systematic derivation of **probability distributions on infinite-dimensional spaces** requires a more formal derivation (introduction of measures etc). This is beyond our scope.



Conclusion on MAP Estimation



By invoking a Bayesian MAP rationale, one can formulate lots of computer vision problem under the form

$$\min_u D(u, f) + \lambda R(u)$$

with an “automated” method for selecting D , R and λ .

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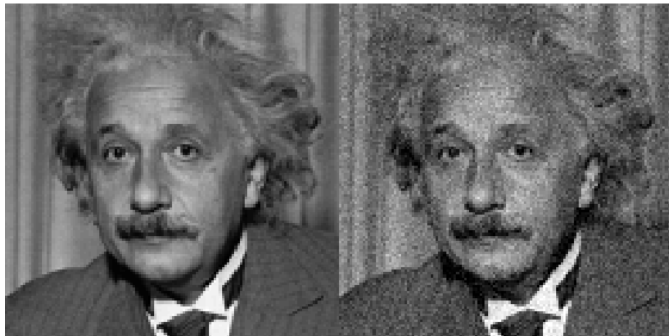
A Few Classic Fidelity Terms

Gaussian noise is fine for modeling “small” perturbations:

$$\mathcal{P}(f|u) \propto \exp\left(-\int \frac{|f(x) - u(x)|^2}{2\sigma^2} dx\right)$$
$$\Rightarrow \min_u \underbrace{\int |f(x) - u(x)|^2 dx}_{:=\|f-u\|_2^2} + \lambda R(u)$$

Ground truth

Denoised

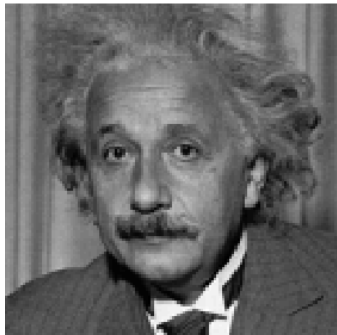


A Few Classic Fidelity Terms

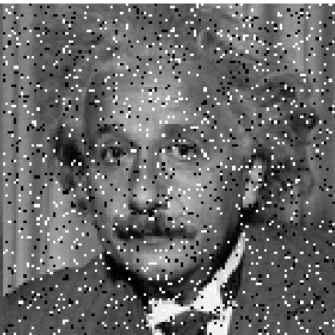
Laplace noise is fine for modeling “impulsive” perturbations:

$$\mathcal{P}(f|u) \propto \exp\left(-\int \frac{|f(x) - u(x)|}{\sigma} dx\right)$$
$$\Rightarrow \min_u \underbrace{\int |f(x) - u(x)| dx}_{:= \|f - u\|_1} + \lambda R(u)$$

Ground truth



Noisy (salt and pepper)

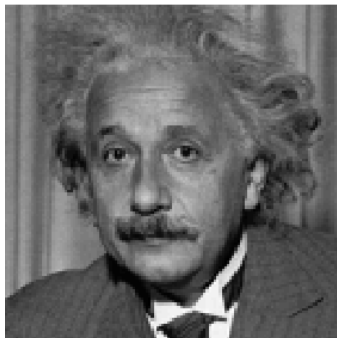


A Few Classic Fidelity Terms

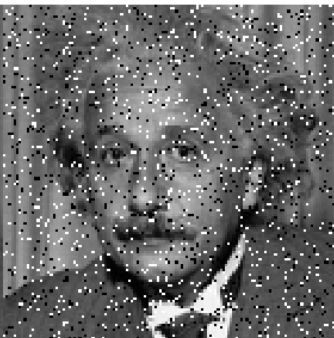
Cauchy noise is even better for modeling “impulsive” perturbations:

$$\mathcal{P}(f|u) \propto \int \frac{1}{|f(x) - u(x)|^2 + \sigma^2} dx$$
$$\Rightarrow \min_u \int \log(|f(x) - u(x)|^2 + \sigma^2) dx + \lambda R(u)$$

Ground truth



Noisy (salt and pepper)

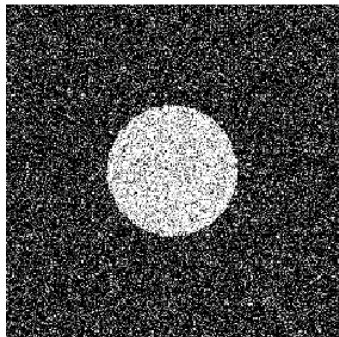


A Few Classic Regularizers

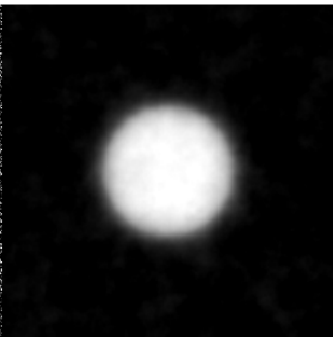
Sobolev regularization tends to favor smoothness:

$$\mathcal{P}(u) \propto \exp \left(-\lambda \int |\nabla u(x)|^2 dx \right)$$
$$\Rightarrow \min_u D(u, f) + \int \|\nabla u(x)\|^2 dx$$

Input (Gaussian)



Denosed (Sobolev)



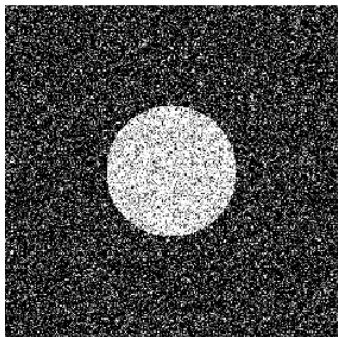
A Few Classic Regularizers

Total variation (TV) tends to favor piecewise constantness:

$$\mathcal{P}(u) \propto \exp\left(-\lambda \int |\nabla u(x)| dx\right)$$

$$\Rightarrow \min_u D(u, f) + \int \|\nabla u(x)\| dx$$

Input (Gaussian)



Denoised (TV)



Why TV Preserves Edges and Sobolev does not

Consider the following step function:

$$u(x) = \begin{cases} 0 & \text{si } x \leq -h \\ \frac{a}{2h}x + \frac{a}{2} & \text{if } x \in [-h, h] \\ a & \text{si } x \geq h \end{cases}$$

Now remark that

$$\int_{\mathbb{R}} |u'(x)|^p dx = 2h^{1-p} a^p$$

to infinity if $p > 1$!!! Means: Sobolev regularization will never ever allow an edge.

But no problem if $p = 1$ (TV) !



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Image Restoration: Motion Blur

Assume the camera lens opens instantly and remains open during the time interval $[0, T]$ in which the camera moves with constant velocity V in x -direction. The observed brightness is

$$g(x, y) = \frac{1}{T} \int_0^T f(x - Vt, y) dt.$$

Inserting $x' \equiv Vt$, we get a convolution

$$g(x, y) = \frac{1}{VT} \int_0^{VT} f(x-x', y) dx' = \int_{-\infty}^{\infty} f(x-x', y-y') h(x', y') dx' dy',$$

with the anisotropic blur kernel:

$$h(x', y') = \frac{1}{VT} \cdot \delta(y') \cdot \chi_{[0, VT]}(x'),$$

and

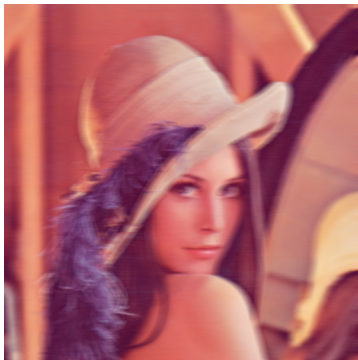
$$\chi_{[a,b]}(x') = \begin{cases} 1, & \text{if } x' \in [a, b] \\ 0, & \text{else} \end{cases} \quad (\text{box filter})$$



Example: Motion Blur



Original



Motion-blurred

(Author: D. Cremers)

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Image Restoration: Defocus Blur

Defocus blur arises with real (in particular thick) lenses because structures are increasingly blurred, the further they are from the focal plane.

Depending on the focal setting and the depth of the scene, we will therefore observe a **space-varying blur** which allows us to infer the local depth (**shape from focus / defocus**).

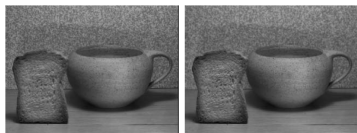


Scene captured with three different focal settings.

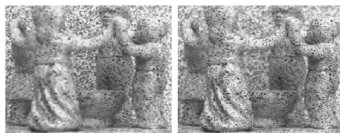
(Source: Favaro, Soatto, PAMI 2005)



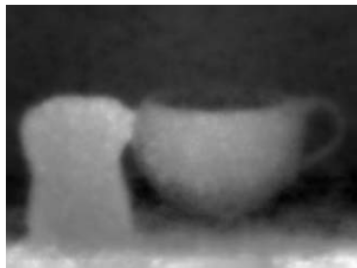
Image Restoration: Defocus Blur



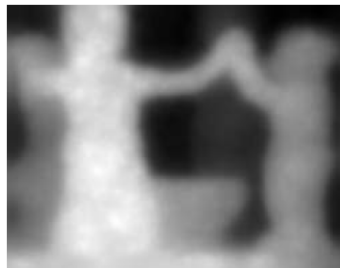
images with different focus



images with different focus



depth reconstruction



depth reconstruction

(Favaro et al., IEEE T. on PAMI 2008)



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Image Restoration: Super Resolution

Super resolution from video exploits the redundancy available in multiple images. We assume that each image f_i is a blurred and downsampled version of a high-resolution scene.

We can try to recover a high-resolution image u with a variational approach of the form:

$$\min_u \sum_{i=1}^n \int |A(u \circ w_i) - f_i| dx + \lambda \int |\nabla u| dx.$$

The deformation field $w_i : \Omega \rightarrow \Omega$ models the warping from the original scene into image i , and A is a linear operator modeling the blurring and downsampling. Again, the variational approach aims at inverting an image formation process of the form:

$$f_i = A(u \circ w_i) + \eta,$$

which states that the observed image is obtained from the “true” image by nonrigid deformation, blurring and downsampling plus additive Poisson-distributed noise η .



Image Restoration: Super Resolution



One of several input images



Superresolution estimate

(Schoenemann, Cremers, IEEE T. on Image Processing 2012)

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Image Restoration: Inpainting

Image inpainting is a particular image restoration technique which explicitly handles (interpolate and / or extrapolate) missing data.

factory for images since it is overly sim
on solving for level lines with minimal cur
an anisotropic diffusion PDE model. The
problem was Nitzberg and Mumford's 2.1-D
Sapiro, Caselles, and Ballester [8] introdu
g through the inpainting domain, but only
n anisotropic diffusion PDE model. The fi
obscuring foreground object. Inpainting is
paining prefers straight contours as they
2], based on a variant of the Mumford-Sh
ed for image denoising by Rudin, Osher,
e of TV regularization was originally dev
round object. Inpainting is an interpolati
ng domain, but only if the length to be r
nal TV, but this is less successful for rec
asing. Inpainting is also used to solve deo



Corrupted

Denoised

Assume $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ a graylevel image, but only $\Omega_D \subset \Omega$ is “reliable”. Then, denoising (or deblurring, etc.) should not use the f -data over $\Omega \setminus \Omega_D$. The standard TV-inpainting model is then:

$$\min_u \int_{\Omega_D} |u - f|^2 dx + \lambda \int_{\Omega} |\nabla u| dx.$$



Limits of TV-inpainting



TV-inpainting is a very naive interpolation technique which does not transport texture...



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Variations around the Inpainting Model

- Compression by Diffusion



- Poisson Image Editing [Perez et al, SIGGRAPH 2003]



How would you do that ?



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