## Variational Methods for Computer Vision: Exercise Sheet 2

Exercise: November 07, 2017

## Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. A multidimensional filter ist called separable, if it can be decomposed in one dimensional filter operations. Prove that the convolution of an image f with a Gaussian kernel K of standard deviation  $\sigma > 0$ ,

$$K(x,y) := \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right),\,$$

can be written as the convolution with two one-dimensional filters:

$$k_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$
 and  $k_2(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$ .

Hence:

$$(f * K)(x, y) = ((f * k_1) * k_2)(x, y),$$

Explain why the separability of a filter is a desirable property.

2. Let  $f \in C^2(\Omega; \mathbb{R})$  with  $\Omega \subset \mathbb{R}^2$  be a real valued function and let  $R \in SO(2)$  be a rotation matrix. Prove that the magnitude of the gradient and the Laplace operator are rotationally covariant by showing the following identities:

(a) 
$$\nabla (f \circ R) = R^{\top} \circ (\nabla f) \circ R$$

(b) 
$$\|\nabla(f \circ R)\| = \|(\nabla f) \circ R\|$$

(c) 
$$\Delta(f \circ R) = (\Delta f) \circ R$$

Reminders:

•  $R \in SO(2)$  denotes  $2 \times 2$  matrices with det(R) = 1 and  $R^TR = RR^T = I$  and can be written as

$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

for some  $\alpha \in [0, 2\pi)$ .

• The multivariate chain-rule for the Jacobian is

$$J_{f \circ g}(a) = J_f(g(a)) \circ J_g(a).$$

• For matrix multiplication and the dot product we have the following identity

$$\langle Ax, y \rangle = (Ax)^{\top} y = x^{\top} A^{\top} y = \langle x, A^{\top} y \rangle.$$

3. The general diffusion equation can be written as follows

$$\begin{split} \partial_t u &= \operatorname{div}(g \cdot \nabla u), & \text{in } \Omega \times [0, \infty), \\ \partial_\nu u &= 0, & \text{on } \partial\Omega \times [0, \infty) \\ u(x, 0) &= u_0(x), & \text{for } x \in \Omega, \end{split}$$

where  $u \in C^2(\Omega \times \mathbb{R}_0^+; \mathbb{R})$  with  $\Omega \subset \mathbb{R}^2$  describes the complete diffusion process and solves the partial differential equation. Note that  $\partial_{\nu} u = \langle \nabla u, \nu \rangle$  is the gradient in normal direction  $\nu$ . Prove the following identities:

(a) linear homogeneous diffusion:

$$\operatorname{div}(g \cdot \nabla u) = g\Delta u, \qquad g \in \mathbb{R}.$$

(b) linear inhomogeneous diffusion:

$$\operatorname{div}(g \cdot \nabla u)(x) = g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x) \rangle, \qquad g \in C^{1}(\Omega; \mathbb{R}).$$

## **Part II: Practical Exercises**

This exercise is to be solved during the tutorial.

- 1. Download the archive file vmcv\_ex02.zip from the homepage and unzip it in your home folder. Use the template file diffusion\_filter.m to implement a nonlinear diffusion filter and complete the missing code in ll. 60. Test the script on the image lena.png. You can use explicit time discretization and the space discretization presented in the lecture.
- 2. Experiment with the different diffusivity variants and step sizes. What do you observe?
- 3. Create a video using the provided script vmcv\_ex02.m and compare the results.