

Variational Methods for Computer Vision: Solution Sheet 1

Exercise: 24 October 2017

Part I: Theory

1. Refresher: Multivariate analysis.

- (a) i. $\nabla f = (x, y)^\top$
ii. $\nabla f = (x^2 + y^2)^{-1/2}(x, y)^\top$
- (b) i. $J = \begin{pmatrix} \cos(\varphi) & -r \sin(\varphi) \\ \sin(\varphi) & r \cos(\varphi) \end{pmatrix}$
ii. $J = \begin{pmatrix} -r \sin(t) \\ r \cos(t) \end{pmatrix}$
- (c) i. $\operatorname{div} f = 0$
ii. $\operatorname{div} f = 2$
- (d) The solutions for the two functions from 1c are:
i. $\operatorname{curl} f = 2$,
ii. $\operatorname{curl} f = 0$.

Proof for the curl of the gradient:

$$\begin{aligned} \operatorname{curl}(\nabla f) &= \operatorname{curl} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \\ &= \frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \frac{\partial f}{\partial y} && \text{(Symmetry of partial derivatives)} \\ &= 0. \end{aligned}$$

- (e) i. Using the coordinate transformation from 1(b)i with $\det J = r$, the area of a disk D_R of radius R can be calculated as

$$\begin{aligned} \iint_{D_R} dx dy &= \int_0^{2\pi} \int_0^R r dr d\varphi \\ &= 2\pi \left[\frac{1}{2} r^2 \right]_0^R \\ &= \pi R^2. \end{aligned}$$

- ii. Using a parametrization like in 1(b)ii, $\gamma_R: [0, 2\pi] \rightarrow \mathbb{R}^2$, $\gamma_R(t) = (R \cos(t), R \sin(t))^\top$ with $\|\gamma_R'\|_2 = R$, the circumference of a circle with radius R can be calculated as

$$\begin{aligned} \int_{\gamma_R} ds &= \int_0^{2\pi} R d\varphi \\ &= 2\pi R. \end{aligned}$$

(f) First calculate the left-hand side of the divergence theorem:

$$\begin{aligned} \iint_{D_R} \operatorname{div} f \, dx \, dy &= \iint_{D_R} 2 \, dx \, dy \\ &= 2\pi R^2. \end{aligned} \quad (\text{Using 1(e)i})$$

For the right-hand side, first calculate the normal vector. The points on the boundary ∂D_R can be characterized by the zero set of $g(x, y) = x^2 + y^2 - R^2$. Calculating the gradient $\nabla g = (2x, 2y)^\top$ will give the direction of the normal n , and normalizing the gradient yields $n = (x^2 + y^2)^{-1/2}(x, y)^\top = (x, y)^\top / R$. Now the integral becomes

$$\begin{aligned} \int_{\partial D_R} \langle f, n \rangle \, ds &= \int_{\gamma_R} \frac{1}{R}(x^2 + y^2) \, ds \\ &= \int_{\gamma_R} R \, ds \\ &= 2\pi R^2, \end{aligned} \quad (\text{Using 1(e)ii})$$

which is equal to the left-hand side.

2. (a) i. Associativity:

$$\begin{aligned} ((f * g) * h)(u) &= \int_{\mathbb{R}} (f * g)(x) h(u - x) \, dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) g(x - y) \, dy \right) h(u - x) \, dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x - y) h(u - x) \, dy \, dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x - y) h(u - x) \, dx \, dy && (\text{Fubini's theorem}) \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x - y) h(u - x) \, dx \, dy \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g((x + y) - y) h(u - (x + y)) \, dx \, dy && (\text{Translation invariance}) \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x) h(u - y - x) \, dx \, dy \\ &= \int_{\mathbb{R}} f(y) (g * h)(u - y) \, dy \\ &= (f * (g * h))(u). \end{aligned}$$

Remark: The translation invariance step can be seen as a special case of

$$\int_S f(s) \, ds = \int_P f(\phi(p)) |\det J_\phi(p)| \, dp,$$

in the following way:

$$\begin{aligned} \int_{\mathbb{R}} g(x - y) \cdot h(u - x) \, dx &= \int_{\mathbb{R}} (g \circ \varphi_1^y)(x) \cdot (h \circ \varphi_2^u)(x) \, dx \\ &= \int_{\mathbb{R}} (g \circ \varphi_1^y)(\varphi_y(x)) \cdot (h \circ \varphi_2^u)(\varphi_y(x)) \underbrace{|\det J_{\varphi_y}|}_{=1} \, dx \\ &= \int_{\mathbb{R}} g(x) h(u - y - x) \, dx, \end{aligned}$$

with $\varphi_y(x) = x + y$, $\varphi_y(\mathbb{R}) = \mathbb{R}$, $\varphi_1^y(x) = x - y$, $\varphi_2^u(x) = u - x$.

ii. Commutativity:

$$\begin{aligned}
 (f * g)(u) &:= \int_{\mathbb{R}} f(x) g(u - x) dx \\
 &= \int_{\mathbb{R}} g(\varphi_u(x)) f(u - \varphi_u(x)) |\det J_{\varphi_u}| dx \\
 &= \int_{\mathbb{R}} f(u - x) g(x) dx \\
 &= \int_{\mathbb{R}} g(x) f(u - x) dx \\
 &=: (g * f)(u),
 \end{aligned}$$

with $\varphi_u(x) = u - x$, $|\det J_{\varphi_u}| = 1$, $\varphi_u(\mathbb{R}) = \mathbb{R}$.

iii. Distributivity:

$$\begin{aligned}
 f * (g + h)(u) &= \int_{\mathbb{R}} f(x)(g + h)(u - x) dx \\
 &= \int_{\mathbb{R}} f(x)g(u - x) + f(x)h(u - x) dx \\
 &= \int_{\mathbb{R}} f(x)g(u - x) dx + \int_{\mathbb{R}} f(x)h(u - x) dx \\
 &= (f * g + f * h)(u).
 \end{aligned}$$

(b) We start with the definition of the Fourier transform:

$$\begin{aligned}
 \mathcal{F}\{f * g\}(\nu) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)g(x - y) dy \right) e^{-2\pi i x \nu} dx \\
 &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x - y) e^{-2\pi i x \nu} dx \right) dy.
 \end{aligned}$$

Introducing the substitution $z = x - y$, $dz = dx$ we arrive at

$$\begin{aligned}
 \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x - y) e^{-2\pi i x \nu} dx \right) dy &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(z) e^{-2\pi i (z+y)\nu} dz \right) dy \\
 &= \int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} \int_{\mathbb{R}} g(z) e^{-2\pi i z \nu} dz dy \\
 &= \underbrace{\int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} dy}_{=: \mathcal{F}\{f\}(\nu)} \underbrace{\int_{\mathbb{R}} g(z) e^{-2\pi i z \nu} dz}_{=: \mathcal{F}\{g\}(\nu)}.
 \end{aligned}$$

As the Fourier transform and its inverse can be implemented to run in $\mathcal{O}(n \log n)$ time, convolutions can be computed efficiently by exploiting this property:

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}.$$

(c) Let us consider the difference quotient

$$\frac{(f * g)(x + t) - (f * g)(x)}{t} = \int_{\mathbb{R}} f(y) \frac{g(x + t - y) - g(x - y)}{t} dy.$$

Now taking the limit $t \rightarrow 0$ we have

$$\begin{aligned}
\frac{d}{dx}(f * g)(x) &= \lim_{t \rightarrow 0} \frac{(f * g)(x + t) - (f * g)(x)}{t} \\
&= \lim_{t \rightarrow 0} \int_{\mathbb{R}} f(y) \frac{g(x + t - y) - g(x - y)}{t} dy \\
&= \int_{\mathbb{R}} \lim_{t \rightarrow 0} f(y) \frac{g(x + t - y) - g(x - y)}{t} dy \\
&= \int_{\mathbb{R}} f(y) \left(\frac{d}{dx} g \right)(x - y) dy && \text{(see below)} \\
&= f * \frac{dg}{dx} = \frac{dg}{dx} * f.
\end{aligned}$$

Remark: In order to interchange integration and limit, one needs some additional conditions to hold (see Lebesgue's dominated convergence theorem). The theorem requires that

$$F_t(y) := f(y) \frac{g(x + t - y) - g(x - y)}{t},$$

converges pointwise to a function $F_t(y) \rightarrow F(y)$, and F_t is dominated by an integrable function \bar{F} in the sense

$$|F_t(y)| \leq \bar{F}(y), \forall t, \forall y.$$

The remaining equality follows from the above and commutativity of convolution:

$$\frac{d}{dx}(f * g) = \frac{d}{dx}(g * f) = g * \frac{df}{dx} = \frac{df}{dx} * g.$$

Remark: To see the following equality

$$\lim_{t \rightarrow 0} \frac{g(x + t - y) - g(x - y)}{t} = \left(\frac{d}{dx} g \right)(x - y)$$

consider the function $\tilde{g}(x) = g(x - y)$. We differentiate using the chain rule as

$$\tilde{g}'(x) = g'(x - y),$$

which we use to arrive at

$$\lim_{t \rightarrow 0} \frac{g(x + t - y) - g(x - y)}{t} = \lim_{t \rightarrow 0} \frac{\tilde{g}(x + t) - \tilde{g}(x)}{t} = \tilde{g}'(x) = g'(x - y).$$