Variational Methods for Computer Vision: Solution Sheet 1

Exercise: 24 October 2017

Part I: Theory

1. **Refresher: Multivariate analysis.**

   (a) i. $\nabla f = (x, y)^\top$

   ii. $\nabla f = (x^2 + y^2)^{-1/2}(x, y)^\top$

   (b) i. $J = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}$

   ii. $J = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix}$

   (c) i. $\operatorname{div} f = 0$

   ii. $\operatorname{div} f = 2$

   (d) The solutions for the two functions from 1c are:

   i. $\operatorname{curl} f = 2$,

   ii. $\operatorname{curl} f = 0$.

   Proof for the curl of the gradient:

$$\begin{aligned}
\operatorname{curl}(\nabla f) &= \operatorname{curl} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y} && \text{(Symmetry of partial derivatives)} \\
&= 0.
\end{aligned}$$

   (e) i. Using the coordinate transformation from 1(b)i with $\det J = r$, the area of a disk D_R of radius R can be calculated as

$$\begin{aligned}
\iint_{D_R} \mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= 2\pi\left[\frac{1}{2}r^2\right]_0^R \\
&= \pi R^2.
\end{aligned}$$

   ii. Using a parametrization like in 1(b)ii, $\gamma_R \colon [0, 2\pi] \to \mathbb{R}^2, \gamma_R(t) = (R\cos(t), R\sin(t))^\top$ with $\|\gamma_R'\|_2 = R$, the circumference of a circle with radius R can be calculated as

$$\begin{aligned}
\int_{\gamma_R} \mathrm{d}s &= \int_0^{2\pi} R\,\mathrm{d}\varphi \\
&= 2\pi R.
\end{aligned}$$

(f) First calculate the left-hand side of the divergence theorem:

$$\begin{aligned}\iint_{D_R} \operatorname{div} f \,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_R} 2\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\pi R^2. && \text{(Using 1(e)i)}\end{aligned}$$

For the right-hand side, first calculate the normal vector. The points on the boundary ∂D_R can be characterized by the zero set of $g(x,y) = x^2 + y^2 - R^2$. Calculating the gradient $\nabla g = (2x, 2y)^\top$ will give the direction of the normal n, and normalizing the gradient yields $n = (x^2+y^2)^{-1/2}(x,y)^\top = (x,y)^\top/R$. Now the integral becomes

$$\begin{aligned}\int_{\partial D_R} \langle f, n\rangle \,\mathrm{d}s &= \int_{\gamma_R} \frac{1}{R}(x^2+y^2)\,\mathrm{d}s \\ &= \int_{\gamma_R} R\,\mathrm{d}s \\ &= 2\pi R^2, && \text{(Using 1(e)ii)}\end{aligned}$$

which is equal to the left-hand side.

Remark: In general, to compute the normal on the boundary of a set, you can use the fact that the gradient is perpenficular to level sets.

(a) Define an implicit representation of your set, which is negative inside the set and positive outside, such that the boundary corresponds to the zero set. In our example we use $g(x,y) = x^2 + y^2 - R^2$.

(b) TODO...

2. (a) i. Associativity:

$$\begin{aligned}((f*g)*h)(u) &= \int_{\mathbb{R}} (f*g)(x)\,h(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right) h(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)h(u-x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)h(u-x)\,\mathrm{d}x\,\mathrm{d}y && \text{(Fubini's theorem)} \\ &= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} g(x-y)h(u-x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} g((x+y)-y)h(u-(x+y))\,\mathrm{d}x\,\mathrm{d}y && \text{(Translation invariance)} \\ &= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} g(x)h(u-y-x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)(g*h)(u-y)\,\mathrm{d}y \\ &= (f*(g*h))(u).\end{aligned}$$

Remark: The translation invariance step can be seen as a special case of

$$\int_S f(s)\,\mathrm{d}s = \int_P f(\phi(p))|\det J_\phi(p)|\,\mathrm{d}p,$$

in the following way:

$$\int_{\mathbb{R}} g(x-y)\cdot h(u-x)\,\mathrm{d}x = \int_{\mathbb{R}} (g\circ\varphi_1^y)(x)\cdot(h\circ\varphi_2^u)(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} (g\circ\varphi_1^y)(\varphi_y(x))\cdot(h\circ\varphi_2^u)(\varphi_y(x))\underbrace{|\det J_{\varphi_y}|}_{=1}\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} g(x)h(u-y-x)\mathrm{d}x,$$

with $\varphi_y(x) = x + y$, $\varphi_y(\mathbb{R}) = \mathbb{R}$, $\varphi_1^y(x) = x - y$, $\varphi_2^u(x) = u - x$.

ii. Commutativity:

$$(f*g)(u) := \int_{\mathbb{R}} f(x)\,g(u-x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} g(\varphi_u(x))f(u-\varphi_u(x))|\det J_{\varphi_u}|\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} f(u-x)\,g(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} g(x)f(u-x)\,\mathrm{d}x$$
$$=: (g*f)(u),$$

with $\varphi_u(x) = u - x$, $|\det J_{\varphi_u}| = 1$, $\varphi_u(\mathbb{R}) = \mathbb{R}$.

iii. Distributivity:

$$f*(g+h)(u) = \int_{\mathbb{R}} f(x)(g+h)(u-x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} f(x)g(u-x) + f(x)h(u-x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} f(x)g(u-x)\,\mathrm{d}x + \int_{\mathbb{R}} f(x)h(u-x)\,\mathrm{d}x$$
$$= (f*g+f*h)(u).$$

(b) We start with the definition of the Fourier transform:

$$\mathcal{F}\{f*g\}(\nu) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right)e^{-2\pi ix\nu}\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} g(x-y)e^{-2\pi ix\nu}\,\mathrm{d}x\right)\mathrm{d}y.$$

Introducing the substitution $z = x - y$, $\mathrm{d}z = \mathrm{d}x$ we arrive at

$$\int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} g(x-y)e^{-2\pi ix\nu}\,\mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} g(z)e^{-2\pi i(z+y)\nu}\,\mathrm{d}z\right)\mathrm{d}y$$
$$= \int_{\mathbb{R}} f(y)e^{-2\pi iy\nu}\int_{\mathbb{R}} g(z)e^{-2\pi iz\nu}\,\mathrm{d}z\,\mathrm{d}y$$
$$= \underbrace{\int_{\mathbb{R}} f(y)e^{-2\pi iy\nu}\,\mathrm{d}y}_{=:\mathcal{F}\{f\}(\nu)}\ \underbrace{\int_{\mathbb{R}} g(z)e^{-2\pi iz\nu}\,\mathrm{d}z}_{=:\mathcal{F}\{g\}(\nu)}.$$

As the Fourier transform and its inverse can be implemented to run in $\mathcal{O}(n\log n)$ time, convolutions can be computed efficiently by exploiting this property:

$$f*g = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\cdot\mathcal{F}\{g\}\}.$$

(c) Let us consider the difference quotient

$$\frac{(f * g)(x+t) - (f * g)(x)}{t} = \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y.$$

Now taking the limit $t \to 0$ we have

$$\begin{aligned}
\frac{d}{dx}(f * g)(x) &= \lim_{t \to 0} \frac{(f * g)(x+t) - (f * g)(x)}{t} \\
&= \lim_{t \to 0} \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \lim_{t \to 0} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y \\
&= \int_{\mathbb{R}} f(y) (\frac{d}{dx} g)(x-y) \, \mathrm{d}y \qquad \text{(see below)} \\
&= f * \frac{dg}{dx} = \frac{dg}{dx} * f.
\end{aligned}$$

Remark: In order to interchange integration and limit, one needs some additional conditions to hold (see Lebesgue's dominated convergence theorem). The theorem requires that

$$F_t(y) := f(y) \frac{g(x+t-y) - g(x-y)}{t},$$

convergences pointwise to a function $F_t(y) \to F(y)$ (1), and F_t is dominated by an integrable function $\bar{F}$ (2) in the sense

$$|F_t(y)| \leq \bar{F}(y), \forall t, \forall y.$$

(1) Pointwise convergence is easy to see, since g is continuously differentiable:

$$\lim_{t \to 0} F_t(y) = f(y) g'(x-y) \,.$$

(2) To find a dominating function, we can use the fact that g is integrable, i.e.

$$\lim_{|x| \to \infty} g(x) = 0 \quad \text{and thus} \quad \lim_{|x| \to \infty} g'(x) = 0 \,.$$

Since g' is continuous and tends to 0 for large $|x|$, there is an M such that $|g'(\xi)| \leq M$ for all $\xi \in \mathbb{R}$. From the mean value theorem, we further have

$$F_t(y) = f(y) g'(\xi) \quad \text{for some} \quad \xi \in [x-y, x-y+t] \,.$$

Thus, $|F_t(y)| \leq M|f(y)|$. Since f is integrable, $M|f|$ is also integrable, so it meets our requirements for the dominating function.

The remaining equality follows from the above and commutativity of convolution:

$$\frac{d}{dx}(f * g) = \frac{d}{dx}(g * f) = g * \frac{df}{dx} = \frac{df}{dx} * g.$$

Remark: To see the following equality

$$\lim_{t \to 0} \frac{g(x+t-y) - g(x-y)}{t} = (\frac{d}{dx} g)(x-y)$$

consider the function $\tilde{g}(x) = g(x-y)$. We differentiate using the chain rule as

$$\tilde{g}'(x) = g'(x-y),$$

which we use to arrive at

$$\lim_{t \to 0} \frac{g(x+t-y) - g(x-y)}{t} = \lim_{t \to 0} \frac{\tilde{g}(x+t) - \tilde{g}(x)}{t} = \tilde{g}'(x) = g'(x-y).$$