Variational Methods for Computer Vision: Solution Sheet 10

Exercise: January 23, 2018

Part I: Theory

1. (a) We use the fact that for two vectors a and b in $\mathbb{R}^n$,

$$|a+b| \leq |a| + |b| \,,$$

and rewrite the energy as

$$E(u) = \int_\Omega (f_1 - f_2)u + \nu|\nabla u| + f_2 \,\mathrm{d}x \,.$$

Then a short calculation shows for functions u, v and $\alpha \in [0,1]$

$$E(\alpha u + (1-\alpha)v) =$$
$$\int_\Omega (f_1 - f_2)(\alpha u + (1-\alpha)v) + \nu|\alpha\nabla u + (1-\alpha)\nabla v| + \alpha f_2 + (1-\alpha)f_2 \,\mathrm{d}x \leq$$
$$\int_\Omega \alpha(f_1 - f_2)u + (1-\alpha)(f_1 - f_2)v + \alpha\nu|\nabla u| + (1-\alpha)\nu|\nabla v| + \alpha f_2 + (1-\alpha)f_2 \,\mathrm{d}x =$$
$$\alpha E(u) + (1-\alpha)E(v).$$

Thus, E is convex in u.

(b) $[0,1] \subset \mathbb{R}$ is a convex set (which you can show directly by considering a convex combination of two elements of $[0,1]$). Therefore it holds for $u, v \in U$, $x \in \Omega$ and $\alpha \in [0,1]$

$$\alpha u(x) + (1-\alpha)v(x) \in [0,1].$$

Since $x \in \Omega$ can be chosen arbitrary, it follows that $\alpha u + (1-\alpha)v \in U$. Therefore U is a convex set.

(c) Let $F(u) = \int_\Omega (f(x) - u(x))^2 \mathrm{d}x$. Then for all $u \in U$

$$\begin{aligned} F(u) &= \int_{f(x)>1} (f(x) - u(x))^2 \mathrm{d}x + \int_{f(x)<0} (f(x) - u(x))^2 \mathrm{d}x + \int_{f(x)\in[0,1]} (f(x) - u(x))^2 \mathrm{d}x \\ &\geq \int_{f(x)>1} (f(x) - 1)^2 \mathrm{d}x + \int_{f(x)<0} f(x)^2 \mathrm{d}x + 0 = F(f_U), \end{aligned}$$

which implies f_U is the global minimum of $F(u)$ and therefore the projection of f onto the convex set U.

(d) Using the result from the previous exercise sheets

$$\frac{\mathrm{d}E}{\mathrm{d}u} = \frac{\partial \mathcal{L}}{\partial u} - \mathrm{div}\frac{\partial \mathcal{L}}{\partial \nabla u} = 0.$$

The partial derivatives are given by:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} &= f_1 - f_2, \\ \frac{\partial \mathcal{L}}{\partial \nabla u} &= \nu \frac{\nabla u}{|\nabla u|}. \end{aligned}$$