Variational Methods for Computer Vision: Exercise Sheet 1

Exercise: October 24, 2018
Prof. Dr. Daniel Cremers, Marvin Eisenberger, Mohammed Brahimi

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. **Refresher: Multivariate analysis.**

   (a) For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, the *gradient* is defined as $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)^\top$. Calculate the gradients of the following functions.

   i. $f\colon \mathbb{R}^2 \to \mathbb{R}, f(x) = \frac{1}{2}\|x\|_2^2$,

   ii. $f\colon \mathbb{R}^2 \to \mathbb{R}, f(x) = \|x\|_2$.

   Are there any points where the gradient is undefined?

   (b) For a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, the *Jacobian matrix* at the point $a \in \mathbb{R}^n$ is defined as

   $$J_f(a) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

   Calculate the Jacobian matrix of the following functions:

   i. $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^2, f(r,\varphi) = (r\cos(\varphi), r\sin(\varphi))^\top$,

   ii. $f\colon \mathbb{R} \to \mathbb{R}^2, f(t) = (r\cos(t), r\sin(t))^\top$.

   (c) For a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the *divergence* is defined as $\operatorname{div} f = \sum_{i=1}^n \partial f_i/\partial x_i$. Calculate the divergence of the following functions:

   i. $f\colon \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = (-y, x)^\top$,

   ii. $f\colon \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = (x, y)^\top$.

   (d) For a function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$, the *curl* is defined as $\operatorname{curl} g = \partial g_2/\partial x - \partial g_1/\partial y$. Calculate the curl of function 1(c)i. Prove that the identity $\operatorname{curl}(\nabla f) = 0$ is true for any twice continuously differentiable function $f \in C^2(R^2;\mathbb{R})$. Verify the identity with your result from 1(a)i.

   (e) When integrating a function $f\colon S \to \mathbb{R}$ over an open subset $S \subset \mathbb{R}^n$ using a parametrization $P \subset \mathbb{R}^n$, $\phi\colon P \to S$, the Jacobian of ϕ has to be taken into account as follows:

   $$\int_S f(s)\,\mathrm{d}s = \int_P f(\phi(p))|\det J_\phi(p)|\,\mathrm{d}p.$$

   For $\gamma : [a,b] \to \mathbb{R}^n$, the line integral over a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ is given by

   $$\int_\gamma f\,\mathrm{d}s = \int_a^b f(\gamma(t))\|\gamma'(t)\|_2\,\mathrm{d}t$$

   i. Calculate the area enclosed by a circle of radius R.

   ii. Calculate the circumference of a circle of radius R.

The results from task 1b might be helpful.

(f) The divergence theorem (a special case of Stokes' theorem) states that an integral of the divergence of a function $f\colon S \to \mathbb{R}^n$ over a subset $S \subset \mathbb{R}^n$ can be replaced by an integral over the boundary ∂S of S:

$$\int_S \operatorname{div} f \,\mathrm{d}s = \int_{\partial S} \langle f, n \rangle \,\mathrm{d}s,$$

where $\langle \cdot, \cdot \rangle$ is the dot product and n the unit vector pointing in the direction normal to the boundary.

Convince yourself that this formula holds using f from task 1(c)ii and with S being a disk of radius R.

2. **Convolutions and the Fourier transform.**

(a) Let $f, g, h \in L^1(\mathbb{R})$ be absolutely integrable functions. Consider the convolution of the functions f and g:

$$(f * g)(x) = \int_{\mathbb{R}} f(y)\, g(x-y) \,\mathrm{d}y.$$

Show the following three algebraic identities:

i. $(f * g) * h = f * (g * h)$

ii. $f * g = g * f$

iii. $f * (g + h) = f * g + f * h$

(b) Let $\mathcal{F}$ denote the Fourier transform operator:

$$\mathcal{F}\{f\} := \hat{f}(\nu) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \nu} \,\mathrm{d}x.$$

Prove that the Fourier transform of the convolution of two functions is the same as the pointwise multiplication of the respective Fourier transforms:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}.$$

What implications does this have for computing the convolution?

(c) Additionally let f and g be continuously differentiable. Show that:

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = \frac{dg}{dx} * f.$$

The results from 2a might be useful.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. Start MATLAB and visualize the vector fields from exercise 1(c)i and 1(c)ii. The commands `help meshgrid` and `help quiver` can be useful for that. Explain the intuition behind divergence-free and curl-free vector fields!

2. Download the archive `vmcv_ex01.zip` and unzip it on your home folder. In there should be a file named `coins.png`. Load the unzipped image using the following command:

   ```
   f=double(imread('coins.png'));
   ```

   Show the image using MATLAB's command:

   ```
   figure; imshow(uint8(f));
   ```

3. Let W and H denote respectively the width and height of the input image f.

   (a) Compute the gradient $\nabla f = (\partial_x^+ f, \partial_y^+ f)^\mathsf{T}$ of the image using the discretization scheme of forward differences:

   $$(\partial_x^+ f)_{i,j} = \begin{cases} f_{i+1,j} - f_{i,j} & \text{if } i < W \\ 0 & i = W. \end{cases}$$

   $$(\partial_y^+ f)_{i,j} = \begin{cases} f_{i,j+1} - f_{i,j} & \text{if } j < H \\ 0 & j = H. \end{cases}$$

   Notice that the boundary values of the gradient are set to zero.

   (b) Compute the gradient without using any for loops this time. Can you tell the difference?

4. (a) Compute the convolution of the image with a Gaussian kernel. In theory, the Gaussian distribution is nonzero everywhere, however in practice we restrict ourself to truncated kernels. Set the radius of the kernel to $r = \text{ceil}(3 \times \sigma)$. The discrete convolution is given as:

   $$g(i,j) = (w * f)(i,j) := \sum_{m=-r}^{r} \sum_{n=-r}^{r} w(m,n)\, f(i-m, j-n).$$

   The discrete truncated Gaussian kernel can be written as follows:

   $$w(m,n) \propto \exp\left(-\frac{m^2+n^2}{2\sigma^2}\right)$$

   In order to stay consistent with the continuous formulation of the Gaussian distribution make sure to normalize the kernel function such that the following holds:

   $$\sum_{m=-r}^{r} \sum_{n=-r}^{r} w(m,n) = 1.$$

   For simplicity you can ignore pixels where the mask goes beyond the edge of the image.

   (b) Find out how to obtain the same result as in (a) without using any loops. To this end, make yourself familiar with the `conv2` command.

   (c) Now compute the gradient (see Ex. 3) of the image using convolution. Make sure to take care of the boundary values.

5. Let f_σ be the input image convolved with a Gaussian kernel of standard deviation σ. Compute the magnitude of the Gradient $|\nabla f_\sigma|$ for different values for σ. What do you observe?