## Variational Methods for Computer Vision: Solution Sheet 1

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## Part I: Theory

1. Refresher: Multivariate analysis.

(a) i. 
$$\nabla f = (x,y)^{\top}$$
  
ii.  $\nabla f = (x^2 + y^2)^{-1/2}(x,y)^{\top}$ 

$$\begin{array}{ll} \text{(b)} & \text{i. } J = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix} \\ & \text{ii. } J = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix} \end{array}$$

(c) i. 
$$\operatorname{div} f = 0$$
  
ii.  $\operatorname{div} f = 2$ 

(d) The solutions for the two functions from 1c are:

i. 
$$\operatorname{curl} f = 2$$
,

ii. 
$$\operatorname{curl} f = 0$$
.

Proof for the curl of the gradient:

$$\begin{aligned} \operatorname{curl}(\nabla f) &= \operatorname{curl}\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right) \\ &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \\ &= 0. \end{aligned}$$

(Symmetry of partial derivatives)

(e) i. Using the coordinate transformation from 1(b)i with  $\det J = r$ , the area of a disk  $D_R$  of radius R can be calculated as

$$\iint_{D_R} dx \, dy = \int_0^{2\pi} \int_0^R r \, dr \, d\varphi$$
$$= 2\pi \left[ \frac{1}{2} r^2 \right]_0^R$$
$$= \pi R^2.$$

ii. Using a parametrization like in 1(b)ii,  $\gamma_R \colon [0,2\pi] \to \mathbb{R}^2$ ,  $\gamma_R(t) = (R\cos(t),R\sin(t))^\top$  with  $\|\gamma_R'\|_2 = R$ , the circumference of a circle with radius R can be calculated as

$$\int_{\gamma_R} \mathrm{d}s = \int_0^{2\pi} R \,\mathrm{d}\varphi$$
$$= 2\pi R.$$

(f) First calculate the left-hand side of the divergence theorem:

$$\iint_{D_R} \operatorname{div} f \, \mathrm{d}x \, \mathrm{d}y = \iint_{D_R} 2 \, \mathrm{d}x \, \mathrm{d}y$$
 
$$= 2\pi R^2. \tag{Using 1(e)i)}$$

For the right-hand side, first calculate the normal vector. The points on the boundary  $\partial D_R$  can be characterized by the zero set of  $g(x,y)=x^2+y^2-R^2$ . Calculating the gradient  $\nabla g=(2x,2y)^{\top}$  will give the direction of the normal n, and normalizing the gradient yields  $n=(x^2+y^2)^{-1/2}(x,y)^{\top}=(x,y)^{\top}/R$ . Now the integral becomes

$$\int_{\partial D_R} \langle f, n \rangle \, \mathrm{d}s = \int_{\gamma_R} \frac{1}{R} (x^2 + y^2) \, \mathrm{d}s$$

$$= \int_{\gamma_R} R \, \mathrm{d}s$$

$$= 2\pi R^2, \qquad \text{(Using 1(e)ii)}$$

which is equal to the left-hand side.

**Remark:** To compute normals on the boundary  $\partial P$  of a set  $P \subset \mathbb{R}^n$ , you can use the fact that the gradient is perpendicular to level sets. Define an implicit representation of P such that the boundary corresponds to the zero set, i.e. define a differentiable function  $f \colon \mathbb{R}^n \to \mathbb{R}$  such that  $P = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \leq 0\}$  and  $\partial P = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0\}$ . The normal at a point  $\mathbf{x} \in \partial P$  corresponds to  $\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$ . It is pointing outwards, since f is negative inside and positive outside of P.

## 2. (a) i. Associativity:

$$((f*g)*h)(u) = \int_{\mathbb{R}} (f*g)(x) h(u-x) dx$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y)g(x-y) dy \right) h(u-x) dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)g(x-y)h(u-x) dy dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)g(x-y)h(u-x) dx dy \qquad \text{(Fubini's theorem)}$$

$$= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x-y)h(u-x) dx dy$$

$$= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g((x+y)-y)h(u-(x+y)) dx dy \qquad \text{(Translation invariance)}$$

$$= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x)h(u-y-x) dx dy$$

$$= \int_{\mathbb{R}} f(y)(g*h)(u-y) dy$$

$$= (f*(g*h))(u).$$

Remark: The translation invariance step can be seen as a special case of

$$\int_{S} f(s) ds = \int_{P} f(\phi(p)) |\det J_{\phi}(p)| dp,$$

in the following way:

$$\int_{\mathbb{R}} g(x-y) \cdot h(u-x) \, \mathrm{d}x = \int_{\mathbb{R}} (g \circ \varphi_1^y)(x) \cdot (h \circ \varphi_2^u)(x) \, \mathrm{d}x$$

$$= \int_{\mathbb{R}} (g \circ \varphi_1^y)(\varphi_y(x)) \cdot (h \circ \varphi_2^u)(\varphi_y(x)) \underbrace{|\det J_{\varphi_y}|}_{=1} \, \mathrm{d}x$$

$$= \int_{\mathbb{R}} g(x)h(u-y-x) \, \mathrm{d}x,$$

with  $\varphi_y(x) = x + y$ ,  $\varphi_y(\mathbb{R}) = \mathbb{R}$ ,  $\varphi_1^y(x) = x - y$ ,  $\varphi_2^u(x) = u - x$ .

ii. Commutativity:

$$(f * g)(u) := \int_{\mathbb{R}} f(x) g(u - x) dx$$

$$= \int_{\mathbb{R}} g(\varphi_u(x)) f(u - \varphi_u(x)) |\det J_{\varphi_u}| dx$$

$$= \int_{\mathbb{R}} f(u - x) g(x) dx$$

$$= \int_{\mathbb{R}} g(x) f(u - x) dx$$

$$=: (g * f)(u),$$

with  $\varphi_u(x) = u - x$ ,  $|\det J_{\varphi_u}| = 1$ ,  $\varphi_u(\mathbb{R}) = \mathbb{R}$ .

iii. Distributivity:

$$f * (g+h)(u) = \int_{\mathbb{R}} f(x)(g+h)(u-x) dx$$
$$= \int_{\mathbb{R}} f(x)g(u-x) + f(x)h(u-x) dx$$
$$= \int_{\mathbb{R}} f(x)g(u-x) dx + \int_{\mathbb{R}} f(x)h(u-x) dx$$
$$= (f * g + f * h)(u).$$

(b) We start with the definition of the Fourier transform:

$$\mathcal{F}\{f * g\}(\nu) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y)g(x-y) \, dy \right) e^{-2\pi i x \nu} \, dx$$
$$= \int_{\mathbb{R}} f(y) \left( \int_{\mathbb{R}} g(x-y)e^{-2\pi i x \nu} \, dx \right) \, dy.$$

Introducing the substitution z = x - y, dz = dx we arrive at

$$\int_{\mathbb{R}} f(y) \left( \int_{\mathbb{R}} g(x-y) e^{-2\pi i x \nu} \, \mathrm{d}x \right) \, \mathrm{d}y = \int_{\mathbb{R}} f(y) \left( \int_{\mathbb{R}} g(z) e^{-2\pi i (z+y)\nu} \, \mathrm{d}z \right) \, \mathrm{d}y$$

$$= \int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} \int_{\mathbb{R}} g(z) e^{-2\pi i z \nu} \, \mathrm{d}z \, \mathrm{d}y$$

$$= \underbrace{\int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} \, \mathrm{d}y}_{=:\mathcal{F}\{f\}(\nu)} \underbrace{\int_{\mathbb{R}} g(z) e^{-2\pi i z \nu} \, \mathrm{d}z}_{=:\mathcal{F}\{g\}(\nu)}.$$

As the Fourier transform and its inverse can be implemented to run in  $\mathcal{O}(n \log n)$  time, convolutions of two images with n pixels each can be computed efficiently in  $\mathcal{O}(n \log n)$  by exploiting this property:

$$f * g = \mathcal{F}^{-1} \{ \mathcal{F} \{ f \} \cdot \mathcal{F} \{ g \} \}.$$

The direct approach of implementing the convolution has runtime  $\mathcal{O}(n^2)$ .

(c) Let us consider the difference quotient

$$\frac{(f * g)(x+t) - (f * g)(x)}{t} = \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y.$$

Now taking the limit  $t \to 0$  we have

$$\begin{split} \frac{d}{dx}(f*g)(x) &= \lim_{t \to 0} \frac{(f*g)(x+t) - (f*g)(x)}{t} \\ &= \lim_{t \to 0} \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y \\ &= \int_{\mathbb{R}} \lim_{t \to 0} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y \quad \text{(see Remark 1)} \\ &= \int_{\mathbb{R}} f(y) (\frac{d}{dx}g)(x-y) \, \mathrm{d}y \qquad \text{(see Remark 2)} \\ &= f*\frac{dg}{dx} = \frac{dg}{dx} *f. \end{split}$$

The remaining equality follows from the above and commutativity of convolution:

$$\frac{d}{dx}(f*g) = \frac{d}{dx}(g*f) = g*\frac{df}{dx} = \frac{df}{dx}*g.$$

**Remark 1:** In order to interchange integration and limit, one needs some additional conditions to hold (see Lebesgue's dominated convergence theorem). The theorem requires that

$$F_t(y) := f(y) \frac{g(x+t-y) - g(x-y)}{t},$$

convergences pointwise to a function  $F_t(y) \to F(y)$  (1), and  $F_t$  is dominated by an integrable function  $\bar{F}$  (2) in the sense

$$|F_t(y)| < \bar{F}(y), \forall t, \forall y.$$

(1) Pointwise convergence is easy to see, since g is continuously differentiable:

$$\lim_{t\to 0} F_t(y) = f(y)g'(x-y) .$$

(2) To find a dominating function, we can use the fact that q is integrable, i.e.

$$\lim_{|x|\to\infty}g(x)=0\quad \text{ and thus }\quad \lim_{|x|\to\infty}g'(x)=0\ .$$

Since g' is continuous and tends to 0 for large |x|, there is an M such that  $|g'(\xi)| \leq M$  for all  $\xi \in \mathbb{R}$ . From the mean value theorem, we further have

$$F_t(y) = f(y)g'(\xi)$$
 for some  $\xi \in [x - y, x - y + t]$ .

Thus,  $|F_t(y)| \le M|f(y)|$ . Since f is integrable, M|f| is also integrable, so it meets our requirements for the dominating function.

## Remark 2: To see the equality

$$\lim_{t\to 0} \frac{g(x+t-y)-g(x-y)}{t} = \left(\frac{d}{dx}g\right)(x-y)$$

note that for any  $z=\tilde{f}(x)$  we have

$$\lim_{t \to 0} \frac{g(\tilde{f}(x) + t) - g(\tilde{f}(x))}{t} = \lim_{t \to 0} \frac{g(z + t) - g(z)}{t} = g'(z) = g'(\tilde{f}(x)).$$