Variational Methods for Computer Vision: Solution Sheet 5

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Part I: Theory

1. (a)

$$\frac{\mathrm{d}E_{\text{Poisson}}}{\mathrm{d}u} = -\operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right) = -2\operatorname{div}(\nabla u - V) = -2\left(\Delta u - \operatorname{div} V\right) = -2\left(\Delta u - \Delta g\right).$$

(b) Define a difference function d by $d(x) := u(x) - g(x)$. Since differentiation is a linear operation, we can write

$$J_u - J_g = J_d\,, \quad \nabla u_c - \nabla g_c = \nabla d_c \quad \text{for} \quad c \in \{r, g, b\}$$

$$\Rightarrow \quad \|J_u - J_g\|_F^2 = \|J_d\|_F^2 = \left\| \begin{pmatrix} \partial_x d_r & \partial_y d_r \\ \partial_x d_g & \partial_y d_g \\ \partial_x d_b & \partial_y d_b \end{pmatrix} \right\|_F^2$$
$$= \sum_{c=r,g,b} (\partial_x d_c)^2 + (\partial_y d_c)^2 = \sum_{c=r,g,b} |\nabla d_c|^2 = \sum_{c=r,g,b} |\nabla u_c - \nabla g_c|^2$$

Now we can easily conclude that

$$E_{\text{P,vec}}(u) = \int_{\Omega_0} \|J_u - J_g\|_F^2 \,\mathrm{d}x = \sum_{c=r,g,b} \int_{\Omega_0} |\nabla u_c - \nabla g_c|^2 \,\mathrm{d}x = \sum_{c=r,g,b} E_{\text{Poisson}}(u_c)$$

2. We can write the energy functional as a sum of two energies

$$E(u) = E_k(u) + \lambda E_{\text{TV}}(\nabla u) \quad \Rightarrow \quad \frac{\mathrm{d}E(u)}{\mathrm{d}u} = \frac{\mathrm{d}E_k(u)}{\mathrm{d}u} + \lambda \frac{\mathrm{d}E_{\text{TV}}(u)}{\mathrm{d}u}$$

where we know the second term of the Gateaux derivative of E from the last exercise sheet:

$$\frac{\mathrm{d}E_{\text{TV}}}{\mathrm{d}u} = -\frac{\partial \mathcal{L}_{\text{TV}}(\nabla u)}{\partial(\nabla u)} = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

so we can focus on computing the first term:

$$\left.\frac{\mathrm{d}E_k}{\mathrm{d}u}\right|_h = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \left(\tfrac{1}{2}(k * (u + \varepsilon h) - f)^2 - \tfrac{1}{2}(k * u - f)^2\right)\mathrm{d}x$$
$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \left(\tfrac{1}{2}(k * u - f + \varepsilon k * h)^2 - \tfrac{1}{2}(k * u - f)^2\right)\mathrm{d}x$$
$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \left(\tfrac{1}{2}(k * u - f)^2 + \varepsilon(k * u - f)(k * h) + \tfrac{1}{2}\varepsilon^2 (k * h)^2 - \tfrac{1}{2}(k * u - f)^2\right)\mathrm{d}x$$
$$= \lim_{\varepsilon\to 0} \int_\Omega \left((k * u - f)(k * h) + \tfrac{1}{2}\varepsilon (k * h)^2\right)\mathrm{d}x = \int_\Omega (k * u - f)(k * h)\mathrm{d}x\,.$$

Now note that for convolution it holds that

$$\int g(x)(k*h)(x)\mathrm{d}x = \iint g(x)k(x-y)h(y)\mathrm{d}y\mathrm{d}x$$
$$= \iint g(x)\bar{k}(y-x)h(y)\mathrm{d}x\mathrm{d}y = \int (g*\bar{k})(y)h(y)\mathrm{d}y = \int (g*\bar{k})(x)h(x)\mathrm{d}x,$$

where we defined $\bar{k}(x) := k(-x)$, so we obtain (by setting $g := k*u-f$)

$$\left.\frac{\mathrm{d}E_k}{\mathrm{d}u}\right|_h = \int_\Omega (k*u-f)(k*h)\mathrm{d}x = \int_\Omega \left((k*u-f)*\bar{k}\right) h\mathrm{d}x$$
$$\Rightarrow \quad \frac{\mathrm{d}E_k(u)}{\mathrm{d}u} = (k*u-f)*\bar{k}$$

Thus the Euler-Lagrange equations for the functional

$$E(u) = \frac{1}{2}\int_\Omega (u*k-f)^2\,\mathrm{d}x + \lambda\int_\Omega |\nabla u|\mathrm{d}x,$$

are given as follows:

$$(u*k-f)*\bar{k} - \lambda\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0, \qquad \text{in } \Omega,$$
$$\left\langle \frac{\nabla u}{|\nabla u|}, n\right\rangle = 0, \qquad \text{on } \partial\Omega.$$