# Variational Methods for Computer Vision: Solution Sheet 6

Exercise: December 5, 2018
Prof. Dr. Daniel Cremers, Marvin Eisenberger, Mohammed Brahimi

## Part I: Theory

1. We start by the directional derivative, as on the last sheets:

$$\begin{aligned}
\left.\frac{\mathrm{d}E(u)}{\mathrm{d}u}\right|_h &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h)-E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla(u+\varepsilon h), A(u+\varepsilon h)) - \mathcal{L}(u,\nabla u, Au)\right)\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla u+\varepsilon\nabla h, Au+\varepsilon Ah) - \mathcal{L}(u,\nabla u, Au)\right)\,\mathrm{d}x\\
&= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial u}h + \frac{\partial\mathcal{L}}{\partial\nabla u}\nabla h + \frac{\partial\mathcal{L}}{\partial Au}(Ah)\right)\,\mathrm{d}x\\
&= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial u}h - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla u}h + A^*\frac{\partial\mathcal{L}}{\partial Au}h\right)\,\mathrm{d}x + \int_{\partial\Omega} h\left\langle\frac{\partial\mathcal{L}}{\partial\nabla u}, n\right\rangle\,\mathrm{d}s
\end{aligned}$$

   Hence the Euler-Lagrange equation is:

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial u} - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla u} + A^*\frac{\partial\mathcal{L}}{\partial Au} = 0, \text{ for } x\in\Omega\\
&\left\langle\frac{\partial\mathcal{L}}{\partial\nabla u}, n\right\rangle = 0, \text{ for } x\in\partial\Omega, n \text{ is the normal.}
\end{aligned}$$

2. (a) The upsampling operator has the dimensions $U \in \mathbb{R}^{(4nm)\times(nm)}$. It replaces each entry $(i,j)$ of the image $I$ by four pixels at $(2i,2j), (2i+1,2j), (2i,2j+1), (2i+1,2j+1)$. This can be modeled as:

$$U = (I_m\otimes u)\otimes(I_n\otimes u), \text{ where } u = \begin{pmatrix}1\\1\end{pmatrix}$$

   In a similar manner we can compute the matrix $A$ as:

$$A = (I_{0.5m}\otimes a)\otimes(I_{0.5n}\otimes a), \text{ for } a = \begin{pmatrix}0.5 & 0.5\\ 0.5 & 0.5\end{pmatrix}$$

   Let $g = \begin{pmatrix}0.2741\\0.4519\\0.2741\end{pmatrix} \in \mathbb{R}^3$ be the kernel vector containing the weights for a discrete Gaussian blurring and let $G_n \in \mathbb{R}^{n\times n}$ and $G_m \in \mathbb{R}^{m\times m}$ be the tridiagonal matrices containing the elements of $g$ on the diagonals, e.g

$$G_n = \begin{pmatrix}0.4519 & 0.2741 & & \\ 0.2741 & 0.4519 & 0.2741 & \\ & \ddots & \ddots & \\ & & 0.2741 & 0.4519\end{pmatrix}$$

Then the blurring matrix is:

$$B = G_m \otimes G_n$$

Note, that this can also be extended to arbitrary kernels $g$. In this case we get some more general band diagonal matrices $G_n, G_m$.

Not at last we have to compute the matrix representation of the shift operator $S_i$. For the sake of simplicity we are only considering constant shifts $(s_i, s_j) \in \mathbb{Z}^2$. Let now $S_n \in \mathbb{R}^{n\times n}$ be the matrix containing only zeros except for ones at the $s_i$'th subdiagonal. For $s_i = 1$ this would be:

$$S_n = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix}$$

If now $S_m \in \mathbb{R}^{m\times m}$ is the same matrix for the shift in the x-direction, then we get:

$$S_i = S_m \otimes S_n$$

An extension to arbitrary shifts can be derived with a little more effort but it is in principal also straightforward, you just need to insert 1 at the right entries to model the mapping from $(i, j)$ to $(i + s_i, j + s_j)$.

(b) We can use the identity from the first exercise to derive the Euler Lagrange equation of $E$. The energy $E$ can be decomposed into the data term and the TV-regularization, such that: $E = E_{\text{data}} + \lambda E_{\text{TV}}$. Now we can compute the Gateaux derivative of both terms individually. We know already, that:

$$\frac{\mathrm{d}E_{\text{TV}}(u)}{\mathrm{d}u} = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Note, that the concatenation of linear operators $ABS_i$ is again simply a linear operator. We therefore get:

$$\frac{\mathrm{d}E_{\text{data}}(u)}{\mathrm{d}u} = (ABS_i)^* \frac{\partial \mathcal{L}}{\partial ABS_i u} = 2\sum_{i=1}^{n} (ABS_i)^*(ABS_i u - Uf_i).$$