

Variational Methods for Computer Vision: Solution Sheet 9

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Part I: Theory

1. Using the property that the delta distribution $\delta(\phi)$ is the derivative of H , we get

$$|\nabla H(\phi)| = |\delta(\phi) \cdot \nabla \phi(x)| = \delta(\phi) |\nabla \phi(x)|.$$

So we can rewrite the Lagrangian in the following form:

$$\begin{aligned}\mathcal{L} &= f_1 \cdot H(\phi) + f_2(1 - H(\phi)) + \nu |\nabla H(\phi)| \\ &= (f_1 - f_2)H(\phi) + \nu \delta(\phi) |\nabla \phi(x)| + f_2.\end{aligned}$$

So

$$\frac{\partial \mathcal{L}}{\partial \nabla \phi} = \frac{\partial}{\partial \nabla \phi} \nu |\nabla \phi| \delta(\phi) = \nu \cdot \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = H'(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| = \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)|.$$

Using both results we can compute

$$\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla \phi} \right) = \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| - \nu \operatorname{div} \left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|} \right).$$

For further simplification we use the property

$$\operatorname{div}(fV) = \langle \nabla f, V \rangle + f \operatorname{div}(V)$$

for a scalar field $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a vector field $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f = \delta(\phi)$ and $V = \frac{\nabla \phi}{|\nabla \phi|}$:

$$\Rightarrow \frac{\partial E}{\partial \phi} = \delta(\phi)(f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi| - \nu \left\langle \nabla(\delta(\phi)), \frac{\nabla \phi}{|\nabla \phi|} \right\rangle - \delta(\phi) \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right).$$

The inner product can be rewritten using the chain rule and $\langle \nabla \phi, \nabla \phi \rangle = |\nabla \phi|^2$:

$$\left\langle \nabla(\delta(\phi)), \frac{\nabla \phi}{|\nabla \phi|} \right\rangle = \delta'(\phi) \left\langle \nabla \phi, \frac{\nabla \phi}{|\nabla \phi|} \right\rangle = \delta'(\phi) \frac{|\nabla \phi|^2}{|\nabla \phi|} = \delta'(\phi) |\nabla \phi|.$$

Thus, the terms $\nu \delta'(\phi) |\nabla \phi|$ cancel and we finally obtain

$$\frac{\partial E}{\partial \phi} = \delta(\phi)(f_1 - f_2) - \nu \delta(\phi) \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right).$$

2.

$$E(\phi) = \int g(x)|\nabla H(\phi(x))|dx = \int g(x)\delta(\phi)|\nabla\phi|dx.$$

With $\frac{\partial E}{\partial\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \operatorname{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi}\right)$ this leads to

$$\frac{\partial\mathcal{L}}{\partial\phi} = \delta'(\phi)g(x)|\nabla\phi|$$

and

$$\frac{\partial\mathcal{L}}{\partial\nabla\phi} = g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla(\phi)|}.$$

Combining both results and further evaluating $\operatorname{div}\left(g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla(\phi)|}\right)$ gives

$$\begin{aligned}\frac{\partial E}{\partial\phi} &= \frac{\partial\mathcal{L}}{\partial\phi} - \operatorname{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi}\right) \\ &= \delta'(\phi)g|\nabla\phi| - \operatorname{div}\left(g\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right) \\ &= \delta'(\phi)g|\nabla\phi| - \delta'(\phi)g\frac{\langle\nabla\phi, \nabla\phi\rangle}{|\nabla\phi|} - \delta(\phi)\operatorname{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right) \\ &= -\delta(\phi)\operatorname{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right).\end{aligned}$$