Variational Methods for Computer Vision: Solution Sheet 10

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Part I: Theory

1. (a) We use the fact that for two vectors a and b in $\mathbb{R}^n$,

$$|a+b| \leq |a| + |b| \,,$$

and rewrite the energy as

$$E(u) = \int_\Omega (f_1 - f_2)u + \nu|\nabla u| + f_2 \,\mathrm{d}x \,.$$

Then a short calculation shows for functions u, v and $\alpha \in [0,1]$

$$\begin{aligned}
&E(\alpha u + (1-\alpha)v) = \\
&\int_\Omega (f_1 - f_2)(\alpha u + (1-\alpha)v) + \nu|\alpha\nabla u + (1-\alpha)\nabla v| + \alpha f_2 + (1-\alpha)f_2 \,\mathrm{d}x \leq \\
&\int_\Omega \alpha(f_1 - f_2)u + (1-\alpha)(f_1-f_2)v + \alpha\nu|\nabla u| + (1-\alpha)\nu|\nabla v| + \alpha f_2 + (1-\alpha)f_2 \,\mathrm{d}x = \\
&\alpha E(u) + (1-\alpha)E(v).
\end{aligned}$$

Thus, E is convex in u.

(b) $[0,1] \subset \mathbb{R}$ is a convex set (which you can show directly by considering a convex combination of two elements of $[0,1]$). Therefore it holds for $u, v \in U$, $x \in \Omega$ and $\alpha \in [0,1]$

$$\alpha u(x) + (1-\alpha)v(x) \in [0,1].$$

Since $x \in \Omega$ can be chosen arbitrary, it follows that $\alpha u + (1-\alpha)v \in U$. Therefore U is a convex set.

Note: In general, it holds that for any domain Ω,

$$U := \{u : \Omega \to C\} \quad \text{convex} \quad \Leftrightarrow \quad C \quad \text{convex} \,.$$

(c) Let $F(u) = \int_\Omega (f(x) - u(x))^2 \mathrm{d}x$. Then for all $u \in U$

$$\begin{aligned}
F(u) &= \int_{f(x)>1} (f(x)-u(x))^2 \mathrm{d}x + \int_{f(x)<0} (f(x)-u(x))^2 \mathrm{d}x + \int_{f(x)\in[0,1]} (f(x)-u(x))^2 \mathrm{d}x \\
&\geq \int_{f(x)>1} (f(x)-1)^2 \mathrm{d}x + \int_{f(x)<0} f(x)^2 \mathrm{d}x + 0 = F(f_U),
\end{aligned}$$

which implies f_U is the global minimum of $F(u)$ and therefore the projection of f onto the convex set U.

(d) Using the result from the previous exercise sheets

$$\frac{\mathrm{d}E}{\mathrm{d}u} = \frac{\partial \mathcal{L}}{\partial u} - \mathrm{div}\frac{\partial \mathcal{L}}{\partial \nabla u} = 0.$$

The partial derivatives are given by:

$$\frac{\partial \mathcal{L}}{\partial u} = f_1 - f_2,$$
$$\frac{\partial \mathcal{L}}{\partial \nabla u} = \nu \frac{\nabla u}{|\nabla u|}.$$