Variational Methods for Computer Vision: Solution Sheet 11

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Part I: Theory

1. For $u_1, u_2 : \Omega \to \mathbb{R}$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned}
E(\alpha u_1 + (1-\alpha)u_2) &= \sup_{\varphi \in \mathcal{K}} \int_\Omega (\alpha u_1 + (1-\alpha)u_2) \operatorname{div} \varphi \, \mathrm{d}x \\
&= \sup_{\varphi \in \mathcal{K}} \left[\alpha \int_\Omega u_1 \, \mathrm{d}x + (1-\alpha) \int_\Omega u_2 \operatorname{div} \varphi \, \mathrm{d}x \right] \\
&\leq \sup_{\varphi \in \mathcal{K}} \left[\alpha \int_\Omega u_1 \, \mathrm{d}x \right] + \sup_{\varphi \in \mathcal{K}} \left[(1-\alpha) \int_\Omega u_2 \operatorname{div} \varphi \, \mathrm{d}x \right] \\
&= \alpha \sup_{\varphi \in \mathcal{K}} \int_\Omega u_1 \, \mathrm{d}x + (1-\alpha) \sup_{\varphi \in \mathcal{K}} \int_\Omega u_2 \operatorname{div} \varphi \, \mathrm{d}x \\
&= \alpha E(u_1) + (1-\alpha) E(u_2).
\end{aligned}$$

2. (a) ∇I is a 2×1 matrix, so its rank can be at most 1. The same obviously applies to $\nabla I^\top$. For the product of two matrices A, B, the rank satisfies the inequality $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$. Hence, $\operatorname{rank}(\nabla I \nabla I^\top) \leq 1$.
Alternatively,

$$\nabla I \nabla I^\top = \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}.$$

If $I_x = 0$ the first row is zero, if $I_y = 0$ the second row is zero, in both cases it is clear that full rank is not possible. If $I_x \neq 0$ and $I_y \neq 0$, the first row and the second row only differ by a factor of I_y/I_x, which means they are linearly dependent and the matrix cannot have full rank.
Third alternative: The determinant is zero.

(b) Solving for the roots of the characteristic polynomial

$$\det \begin{bmatrix} I_x^2 - \lambda & I_x I_y \\ I_x I_y & I_y^2 - \lambda \end{bmatrix}$$

yields the eigenvalues $\lambda_1 = |\nabla I|^2, \lambda_2 = 0$. For the corresponding eigenvectors v_i, the linear system

$$\begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix} v_i = \lambda_i v_i$$

has be solved. The resulting eigenvectors are $v_1 = \nabla I$ and $v_2 = [-I_y, I_x]^T$ or multiples thereof. This means the eigenvectors are parallel and perpendicular to the image gradient.

3. The Euler-Lagrange equation for the ith component ($i = 1, 2$) reads

$$\frac{\partial E}{\partial v_i} = \frac{\partial \mathcal{L}}{\partial v_i} - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla v_i} = 0.$$

With $\mathcal{L} = (I_x v_1 + I_y v_2 + I_t)^2 + \alpha(|\nabla v_1|^2 + |\nabla v_2|^2)$ one obtains

$$\frac{\partial \mathcal{L}}{\partial v_i} = 2I_i(I_x v_1 + I_y v_2 + I_t)$$

and

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \nabla v} &= 2\alpha \nabla v_i, \\ \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla v_i} &= 2\alpha \Delta v_i, \end{aligned}$$

so the resulting equation is

$$I_i(I_x v_1 + I_y v_2 + I_t) - \alpha \Delta v_i = 0.$$

For a stationary point, the equation must be satisfied for both $i = 1$ and $i = 2$ simultaneously.