Chapter 1

Why Convex Optimization

1.1 Introduction

Optimization problems arise naturally in many computer vision and machine learning applications that estimate pixel values, motions, shapes, or model parameters from input images, videos, range sensors, or training data. By formalizing the problem into a concise mathematical form, we obtain an optimization problem whose solution are the model parameters that best fit the observed data and our prior knowledge of the physical world. The next step, finding a solution to the mathematical model, is far from trivial.

The bitter truth is that most optimization problems are unsolvable. Among the solvable ones, convex problems form a large subset that builds on solid mathematical properties and can be solved efficiently with algorithms that exploit these properties. Most commercial packages for optimization, however, use minimal assumptions on the structure of the optimization in order to fit a large class of problems, albeit in a poor manner, and lead to poor optimization strategies. The goal of this course is to present techniques that exploit the properties of convex optimization problems to develop efficient algorithms for a large set of computer vision and machine learning problems.

In many of these applications the process of creating a model takes a considerable amount of time and effort. Therefore, it is important to understand the properties of the model and the computational consequences of each decision. Very often we have to choose between a *good* model, which we cannot solve and a *bad* model, which can be solved efficiently. To distinguish between the two, it is necessary to be aware of some theory that explains what we can and what we cannot do with optimization problems, and how convexity plays a key role on the solvability of a problem.

This first chapter is a summary of Chapter 1 of *Introductory Lectures on Convex Optimization*, by Nesterov.

1.2 Limitations in General Optimization

Let us start by describing our optimization problem. Let $u \in \mathbb{R}^n$ be an n-dimensional real vector, $C \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$, and E be a real-valued functions of u. We study different variants of the following general minimization problem:

$$\hat{u} \in \arg\min_{u \in C} E(u) \tag{1.1}$$

The function $E : \mathbb{R}^n \to \mathbb{R}$ is the objective function, while the set C is the feasible set. We consider a minimization problem by convention, but we can also consider a maximization problem with $-E$ as objective function.

There is a natural classification of the types of minimization problems that we will study: unconstrained problems where $C = \mathbb{R}^n$, smooth problems where E is differentiable, and non-smooth problems where E is not differentiable. We also distinguish two different types of solutions to the minimization problem.

Definition Global Minimum. $u^\star$ is a global solution of (**??**) if $E(u^\star) \geq E(u)$ for all $u \in C$.

Definition Local Minimum. $u^\star$ is a local solution of (??) if there exists a $r > 0$ such that

$$E(u^\star) \geq E(u) \quad \forall u \in C,\ \|u - u^\star\| < r.$$

Local minima are easier to find that global ones. For instance, given an estimate of the the minimizer u^0, we can create a sequence $\{u^k\}$ that decreases the value of the energy at each step to find the local minimum. Formally, we say that these type of optimization methods create a relaxation sequence $\{E(u^k)\}$ that satisfies $E(u^{k+1}) \leq E(u^k)$ and always improves the initial value of the objective function. If E is bounded below on $\mathbb{R}^n$, then the sequence $\{E(u^k)\}$ converges to a local minimum. Let us formalize what we mean by convergence.

Definition We say that a sequence $\{a^k\} \subset \mathbb{R}^n$ converges to $\hat{a} \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists an $k_0 \in \mathbb{N}$ such that

$$\|a^k - \hat{a}\| < \epsilon \quad \forall k \geq k_0.$$

To implement the idea of relaxation we use another fundamental principle of numerical analysis, the approximation. The approximation replaces the original objective function E by a simplified objective function that is close to the original. When the function is differentiable, we usually resort to local approximations of the objective function based on its Taylor expansion at the current estimate to create linear and quadratic approximations of the objective.

Let $E(u)$ be differentiable at u^0, then for $u \in \mathbb{R}^n$, we have

$$E(u) = E(u^0) + \langle \nabla E(u^0), u - u^0 \rangle + o(\|u - u^0\|) \ \text{ where } \ \lim_{r \to 0} \frac{o(r)}{r} = 0.$$

Function $E(u; u^0) = E(u^0) + \langle \nabla E(u^0), u - u^0 \rangle$ is a linear approximation of E in a neighborhood of u^0. Given an initial estimate of the minimizer u^0, we can then use this linear approximation to reduce the value of $E(u)$ in a neighborhood of u^0. In particular we can decide to iteratively step in the direction of maximum descent as follows:

$$\begin{aligned} u^1 &= u^0 - \tau \nabla E(u^0) \\ u^2 &= u^1 - \tau \nabla E(u^1) \\ &\cdots \\ u^{k+1} &= u^k - \tau \nabla E(u^k). \end{aligned}$$

This gives us a very simple algorithm know as gradient descent. We will see in this course that under certain conditions, the algorithm creates a relaxation sequence that decreases the value of the objective function and converges to a point $\hat{u} \in \mathbb{R}^n$. This a point then satisfies $\hat{u} = \hat{u} - \tau \nabla E(\hat{u}) \Rightarrow \nabla E(\hat{u}) = 0$. This is a necessary condition for optimality, as the next theorem shows.

Theorem 1. First-order Optimality Condition. *Let $u^\star$ be a local minimum of differentiable function $E(u)$. Then $\nabla E(u^\star) = 0$.*

Proof. Since $u^\star$ is a local minimum of $E(u)$, then there exists $r > 0$ such that for all v with $\|v - u^\star\| \leq r$, we have $E(v) \geq E(u^\star)$. Since E is differentiable, this implies that

$$E(v) = E(u^\star) + \langle \nabla E(u^\star), v - u^\star \rangle + o(\|v - u^\star\|) \geq E(u^\star).$$

Thus, for all s we have $\langle \nabla E(u^\star), s \rangle \geq 0$. If we consider the directions s and $-s$, we get $\nabla E(u^\star) = 0$. □

Note that we have proved only a necessary condition of a local minimum. The points satisfying this condition are called the stationary points of function. In order to see that such points are not always the local minima, it is enough to look at function $E(u) = u^3$. The optimality condition $E'(u) = 3u^2 = 0$ suggests that 0 should be a local minimum, even though the function is decreasing for any $u < 0$ and can thus not have a minimum at 0. The point 0 is in fact a stationary point, not a maximum or minimum.

To discern between local minima and stationary points of a function, let us introduce the second-order approximation. Let function $E(u)$ be twice differentiable with Hessian $\nabla^2 E(u)$ at u. Then

$$E(v) = E(u) + \langle \nabla E(u), v - u\rangle + \frac{1}{2}\langle \nabla^2 E(u)(v-u), v-u\rangle + o(\|v-u\|^2).$$

The function $E(v;u) = E(u) + \langle \nabla E(u), v-u\rangle + \frac{1}{2}\langle \nabla^2 E(u)(v-u), v-u\rangle$ is the quadratic (or second-order) approximation of function E at u. Note that the Hessian is a symmetric matrix that can be seen as a derivative of the vector function ∇E. As a result, using a linear approximation to each component of ∇E, we have

$$\nabla E(v) = \nabla E(u) + \nabla^2 E(u)(v-u) + o(\|v-u\|).$$

Using the second-order approximation, we can write down the second- order optimality conditions.

Theorem 2. Second-order Pptimality Condition *Let $u^\star$ be a local minimum of twice differentiahte function $E(u)$. Then $\nabla E(u^\star) = 0$ and $\nabla^2 E(u^\star)$ is symmetric and positive semi-definite, that we denote by $\nabla^2 E(u^\star) \succeq 0$.*

Proof. Since $u^\star$ is a local minimum of function E, there exists $r > 0$ such that

$$E(u) \geq E(u^\star) \quad \forall u \quad \text{with} \quad \|u - u^\star\| < r.$$

The first order optimality condition gives us $\nabla E(u^\star) = 0$ and, as a result

$$E(u) = E(u^\star) + \langle \nabla^2 E(u^\star)(v-u^\star), v-u^\star\rangle + o(\|y-u^\star\|^2) \geq E(u^\star).$$

Thus, $\langle \nabla^2 E(u^\star)(v-u^\star), v-u^\star\rangle \geq 0$. Letting $s = v-u$ we have $\langle \nabla^2 E(u^\star)s, s\rangle \geq 0$, which implies positive semi-definiteness. □

This second-order characteristic of a local minimum is also sufficient.

Theorem 3. *Let function $E(u)$ be twice differentiable on $\mathbb{R}^n$ and let $u^\star$ satisfy $\nabla E(u^\star) = 0$ and $\nabla^2 E(u^\star) \succ 0$. Then $u^\star$ is a strict local minimum of E.*

Proof. In a small neighborhood of $u^\star$, $E(u)$ can be represented as

$$E(u) = E(u^\star) + \langle \nabla^2 E(u^\star)(u-u^\star), u-u^\star\rangle + o(\|u-u^\star\|^2).$$

Since $\lim_{r\to 0} \frac{o(r)}{r} = 0$, there exists a value $\bar{r}$ such that for all $r \in [0, \bar{r}]$ we have

$$|o(r)| \leq \frac{r}{4}\lambda_1,$$

where $\lambda_1 > 0$ is the smallest eigenvalue of matrix $\nabla^2 E(u^\star)$. As $\nabla^2 E(u^\star)$ is symmetric and positive definite, it has positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ and orthogonal eigenvectors $q_1, q_2, \ldots q_n$, such that $\nabla^2 E(u^\star) = \sum_{1\leq i\leq n} \lambda_i q_i^T q_i$ and $\|q_i^T v\| = \|v\|$ for all $v \in \mathbb{R}^n$. As a result,

$$E(u) \geq E(u^\star) + \frac{\lambda_1}{2}\|u-u^\star\|^2 + o(\|u-u^\star\|^2) \geq E(u^\star) + \frac{\lambda_1}{4}\|u-u^\star\|^2 \geq E(u^\star). \tag{1.2}$$

□

For general optimization problems, we thus require second-order differentiablity to formulate necessary and sufficient optimality conditions. The optima described by these conditions is, moreover, only local. This is quite disappointing because most applications in computer vision and machine learning have objective functions that are not differentiable, where these general optimality conditions are meaningless. Even in the rare cases where second-order derivatives exists, computing the Hessian is not feasible because the size of the problem is too large. For these reasons, we resort to the field of convex optimization. Convex optimization is a fairyland where the objective function does not need to be differentiable, optimality conditions are not only necessary but sufficient, and the algorithms scale well with the size of the problem.

Chapter 2

Convex Analysis

2.1 Convex Optimization

We start this section with the unconstrained minimization problem

$$\min_{u \in \mathbb{R}^n} E(u). \tag{2.1}$$

In the general situation we cannot do too much: even when the function is smooth, the gradient method converges only to a stationary point of function E and second-order differentiablity is necessary to derive optimality conditions for a local minimum that are necessary and sufficient. To make the problem tractable we introduce a key assumption on the kind of functions E that we minimize. In particular, we call for the following property: for any E differentiable, the first-order optimality condition should necessary and **sufficient** for a point to be a **global solution** of (**??**). Convex functions come with this guarantee.

Definition A function $E\colon \mathbb{R}^n \to \mathbf{R}$ is convex if and only if for any $u, v \in \mathbb{R}^n$ and $\theta \in [0, 1]$

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v).$$

E is strictly convex if the inequality is strict for all $\theta \in (0, 1)$, $v \neq u$.

The definition of convex functions implicitly assumes that it is possible to evaluate the function at any point of the segment

$$[u, v] = \{z = \theta u + (1-\theta)v \colon 0 \leq \theta \leq 1\}.$$

As a result, it is natural to consider a set that contains the whole segment between any two points in the set. Such sets are called convex.

Definition Convex Sets. The set C is convex if for any $u, v \in C$ and $\theta \in [0, 1]$, $\theta u + (1-\theta)v \in C$.

We can then include this notion in the definition of convex functions with restricted domain.

Definition The **domain** of a function $E\colon \mathbb{R}^n \to \mathbb{R}$ is the set

$$\mathrm{dom}(E) = \{u \in \mathbb{R}^n \colon E(u) < \infty\}$$

We can now extend the definition of convexity to functions.

Definition Convex Function. The function $E : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is convex if

- its domain $\mathrm{dom}(E)$ is a convex set.
- For all $u, v \in \mathrm{dom}(E)$ and all $\theta \in [0,1]$ it holds that

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v).$$

E is **strictly convex** if the inequality is strict for all $\theta \in (0,1)$, $v \neq u$.

In the following we assume that the domain of E is not empty, that is, the function E is proper.

Definition Function $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ is **proper** if its domain is not empty.

This course will investigate convex minimization problems, they are characterized by the form

$$\hat{u} \in \arg\min_{u \in C} E(u), \tag{2.2}$$

where C is a convex set and E is a convex function. To write such a problem in our familiar unconstrained optimization form, we we define the **extended real-valued function** $\tilde{E}$ by introducing the constraint $u \in C$ into the domain of the original energy function E:

$$\tilde{E} : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \qquad \tilde{E}(u) = \begin{cases} E(u) & \text{if } u \in C, \\ \infty & \text{else.} \end{cases}$$

We can then re-write (**??**) as

$$\hat{u} \in \arg\min_{u \in \mathbb{R}^n} \tilde{E}(u).$$

2.2 Convex Sets

We have already seen some convex sets as a result of convex functions

Lemma 4. *If E is a convex function, then for any $\beta \in \mathbb{R}$, its level set $\{u : E(u) \leq \beta\}$ is either convex or empty.*

Proof. Let $u, v \in \mathrm{dom}(E)$ with $E(u) \leq \beta$ and $E(v) \leq \beta$, by convexity of E we have $\theta u + (1-\theta)v \in \mathrm{dom}(E)$ and

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v) \leq \theta\beta + (1-\theta)\beta = \beta.$$

□

Lemma 5. *Let E be a convex function, then its **epigraph** $epi(E) = \{(u, \beta) : E(u) \leq \beta\}$ is a convex set.*

Proof. Let $(u, \alpha), (v, \beta) \in \mathrm{epi}(E)$, then $u, v \in \mathrm{dom}(E)$ with $E(u) \leq \alpha$ and $E(v) \leq \beta$, by convexity of E we have $\theta u + (1-\theta)v \in \mathrm{dom}(E)$ and $\theta(u,\alpha) + (1-\theta)(v,\beta) \in \mathrm{epi}(E)$ because

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v) \leq \theta\alpha + (1-\theta)\beta.$$

□

To determine if a set is convex, a few properties are useful.

Lemma 6. *Let $C \subset \mathbb{R}^n, D \subset \mathbb{R}^m$ be convex sets and $\mathscr{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator, then the following sets are convex*

- *Intersection* $C \cap D$.
- *Sum* $C + D = \{u = x + y \ : \ x \in C, \ y \in D\}$ *if* $n = m$.
- *Affine image* $\mathscr{A}(C) = \{u \in \mathbb{R}^m \ : \ u = \mathscr{A}(x), \ x \in C\}$
- *Inverse affine image* $\mathscr{A}^{-1}(D) = \{v \in \mathbb{R}^n \ : \ \mathscr{A}(v) \in D\}$

Proof. Left as exercise □

As a result of the previous lemma, the following sets are convex

- Half-space $\{u \in \mathbb{R}^n \ : \ \langle a, u\rangle \leq \beta\}$ is convex since linear functions are convex.
- Polytope $\{u \in \mathbb{R}^n \ : \ \langle a_i, u\rangle \leq b_i\}$ is convex as an intersection of convex sets.
- Ellipsoid $\{u \in \mathbb{R}^n \ : \ \langle Au, u\rangle \leq 1 \text{ with } A \succeq 0\}$ because the function $\langle Au, u\rangle$ is a convex function.

2.3 Convex Functions

In order to determine if a function is convex, it is useful to know some equivalent definitions of convexity.

Theorem 7. Convexity and Epigraphs. *A proper function* $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is convex if and only if its epigraph is convex.*

Proof. We have already seen one direction, the other is an exercise. □

Lemma 8. Jensen's Inequality. *For any convex function* E*,* $u_1, \dots, u_m \in$ *dom*(E) *and coefficients* $\theta_1, \dots, \theta_m \geq 0$ *such that* $\sum_{i=1}^m \theta_i u_i = 1$ *it holds*

$$E(\sum_{i=1}^m \theta_i u_i) \leq \sum_{i=1}^m \theta_i E(u_i)$$

.

Proof. By induction on m. The case $m = 2$ is a result of the definition and the general an exercise. □

Corollary 9. *For any* u *a convex combination of* $u_1, \dots, u_m \in$ *dom*(E)*,* $E(u) \leq \max_{1 \leq i \leq m} E(u_i)$*.*

Corollary 10. *Let* $\Delta = \operatorname{Conv}\{u_1, \dots, u_m\}$ *be the convex hull of* $u_1, \dots, u_m$*, then*

$$\max_{u \in \Delta} E(u) = \max_{1 \leq i \leq m} E(u_i).$$

Lemma 11. *Function* $E : C \to \mathbb{R}$ *is convex if and only if* C *is convex and for all* $u, v \in C$*,* $\beta \geq 0$ *such that* $u + \beta(u - v) \in C$ *it holds that*

$$E(u + \beta(u - v)) \geq E(u) + \beta(E(u) - E(v)).$$

Proof. Let E be convex, we first prove the alternative definition. Given $\beta > 0$ define $\theta = \frac{\beta}{\beta+1} \in (0, 1]$ and $x = u + \beta(u - v)$ such that

$$u = \frac{1}{1+\beta}(x + \beta v) = (1 - \theta)x + \theta v$$

by convexity of E,

$$E(u) \leq (1 - \theta)E(x) + \theta E(v) = \frac{1}{1+\beta}E(u + \beta(u - v)) + \frac{\beta}{1+\beta}E(v)$$
$$(1 + \beta)E(u) - \beta E(v) \leq E(u + \beta(u - v))$$

Let us now prove that this alternative definition implies convexity. Given any $u, v \in \mathrm{dom}(E)$, $\theta \in (0, 1]$, define $\beta = \frac{1-\theta}{\theta}$ and $x = \theta u + (1-\theta)v$ such that

$$u = \frac{1}{\theta}(x - (1-\theta)v) = x + \beta(x - v)$$

the inequality reads

$$\begin{aligned}
&E(u) = E(x + \beta(x - v)) \geq E(x) + \beta[E(x) - E(v)] \\
&E(u) \geq (1+\beta)E(x) - \beta E(v) = \frac{1}{\theta}E(x) - \frac{1-\theta}{\theta}E(v) \\
&\theta E(u) + (1-\theta)E(v) \geq E(\theta u + (1-\theta)v)
\end{aligned}$$

□

Theorem 12. Monotonicity of the gradient *Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and continuously differentiable, then E is convex if and only if for any $u, v \in dom(E)$*

$$E(v) \geq E(u) + \langle \nabla E(u), v - u \rangle.$$

Proof. Given $u, v \in \mathrm{dom}(E)$, and $\theta \in [0, 1]$, let $u_\theta = \theta u + (1-\theta)v$. If E is continuously differentiable and satisfies the theorem's inequality, we have

$$\begin{aligned}
E(u_\theta) &\geq E(v) + \langle \nabla E(u_\theta), v - u_\theta \rangle = E(v) + \theta\langle \nabla E(u_\theta), v - u \rangle \\
E(u_\theta) &\geq E(u) + \langle \nabla E(u_\theta), u - u_\theta \rangle = E(u) - (1-\theta)\langle \nabla E(u_\theta), v - u \rangle.
\end{aligned}$$

Multiplying the first inequality by $1-\theta$, the second by θ, and adding the results, we get the inequality that defines a convex function $E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$.
We now prove that a convex and continuously differentiable function satisfies the theorem's inequality. Given $u, v \in \mathrm{dom}(E)$, as E is convex for any $\theta \in [0, 1]$

$$E(v) \geq \frac{1}{1-\theta}[E(u_\theta) - \theta E(u)] = E(u) + \frac{1}{1-\theta}[E(u_\theta) - E(u)] = E(u) + \frac{1}{1-\theta}[E(\theta u + (1-\theta)v) - \theta E(u)]. \tag{2.3}$$

As E is differentiable, the limit when θ tends to 1 exists and we get $E(v) \geq E(u) + \langle \nabla E(u), v - u \rangle$. □

2.3.1 Necessary and Sufficient Optimality Conditions

Theorem 13. *Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex. Any local minimum of E is global.*

Proof. Let $u^\star$ be a global minimum of E and $\bar{u}$ a local minimum that is not global, that is, $E(u^\star) < E(\bar{u})$. By definition of local minimum, there exists an $\epsilon > 0$ such that $E(v) \geq E(\bar{u})$ for any $v \in \mathrm{dom}(E)$ with $\|\bar{u} - v\| < \epsilon$. As $u^\star, \bar{u} \in \mathrm{dom}(E)$ convex, $\theta\bar{u} + (1-\theta)u^\star \in \mathrm{dom}(E)$ and

$$E(\theta\bar{u} + (1-\theta)u^\star) \leq \theta E(u^\star) + (1-\theta)E(\bar{u}) < E(\bar{u})$$

As θ tends to 1, $\|\theta\bar{u} + (1-\theta)u^\star - \bar{u}\| < \epsilon$ and this contradicts the definition of $\bar{u}$ as local minimum. □

When the function is differentiable, we can now prove that first-order optimality conditions are sufficient.

Theorem 14. *If $E \colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable function with $\nabla E(u^\star) = 0$ then $u^\star$ is the global minimum of $E(x)$.*

Proof. As $\nabla E(u^\star) = 0$, the inequality $E(v) \geq E(u^\star) + \langle \nabla E(u^\star), v - u^\star \rangle$ $\forall v \in \mathrm{dom}(E)$ gives us the condition $E(v) \geq E(u^\star)$ that characterizes a global minimum. □

When the objective function is two-times differentiable, we can also characterize convexity in terms of the Hessian.

Theorem 15. *Two times continuously differentiable function $E \colon \mathbb{R}^n \to \mathbb{R}$ is convex if and only for any $u \in \mathbb{R}^n$ we have $\nabla^2 E(u) \succeq 0$.*

Proof. This is part of an exercise sheet. □

As a result, for any matrix A symmetric and positive semi-definite, the quadratic function $E(u) = \alpha + \langle a, u \rangle + \langle a, Au \rangle$ is convex because $\nabla^2 E(u) = A \succeq 0$.

2.3.2 Analytic Properties of Convex Functions

The behavior of convex functions at the boundary of their domain can be out of control. To prevent this case, we ask the functions to be closed.

Definition Closed convex function. A convex function is closed if its epigraph is closed.

Lemma 16. *If E is convex and closed, all its level sets are closed.*

Proof. For each β, the level-set $\{u \,:\, E(u) = \beta\} = \mathrm{epi}(E) \cap \{(x,t) \,:\, t = \beta\}$ can be described as the intersection of the epigraph of E, which is closed and convex, and the closed and convex set $\{(x,t) \,:\, t = \beta\}$. □

If E is convex and continuous and its domain $\mathrm{dom}(E)$ is closed, then E is closed. The converse is not true, a closed convex function is not necessarily continuous. Consider the following examples:

- $E(u) = \frac{1}{u}$ is convex, has an open domain $\mathrm{dom}(E) = \mathbb{R}_{++} = \{u \in \mathbb{R} \,:\, u > 0\}$, but is closed because its epigraph $\{(u,t) \in \mathbb{R} \times \mathbb{R}_{++} \,:\, \frac{1}{t} \leq u\}$ is closed.
- Function $E(u) = \|u\|$, where $\|\cdot\|$ is any norm, is closed and convex as a result of the triangle inequality and homogeneity properties that define any norm:

  $$\|\theta u + (1-\theta)v\| \leq \|\theta u\| + \|(1-\theta)\|v\| = |\theta|\|u\| + |1-\theta|\|\|v\| = \theta\|u\| + (1-\theta)\|\|v\|$$

  The norms more common in computer vision and machine learning are the ℓ_p norms:

  $$\|u\| = \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}} \quad u \in \mathbb{R}^n.$$

  - the Euclidean norm: $|u| = \sqrt{\sum_{i=1}^{n} u_i^2}$.
  - the non-differentiable ℓ_1 norm $\|u\|_1 = \sum_{i=1}^{n} |u_i|$.
  - the ℓ_∞ norm $\|u\|_\infty = \max_{1 \leq i \leq n} \ |u_i|$.

  Any norm defines a system of balls $B_p(u,r) = \}v \in \mathbf{R^n} \,:\, \|v-u\|_p \leq r\{$ that are convex.

- the function

  $$E(x,y) = \begin{cases} 0 & \text{if } \ x^2 + y^2 < 1 \\ \phi(x,y) & \text{if } \ x^2 + y^2 = 1 \end{cases}$$

  with domain the unit ball is closed and convex for any $\phi(x,y) > 0$ defined on the unit circle, the boundary of the function domain. Imposing that the function is closed, which implies $\phi(x,y) = 0$, ensures that the function is well-behaved also on the boundary of its domain.

The behavior of convex function at the boundary of their domain can be disappointing, but their behavior in the interior of its domain is very simple.

Theorem 17. *Let function $E\colon C \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex, then E is locally bounded at $u \in \mathrm{int}dom(E)$.*

Proof. Let us choose $\epsilon > 0$ such that $u \pm \epsilon e_i \in \mathrm{int\ dom}(E) \quad i = 1, \dots, n$, where e_i is the i-th coordinate vector of $\mathbb{R}^n$ and define $\hat{\epsilon} = \frac{\epsilon}{\sqrt{n}}$. A simple drawing show us that

$$B(u, \hat{\epsilon}) \subset \Delta = \mathrm{Conv}\{u \pm \epsilon e_i \quad i = 1, \dots, n\}.$$

From the corollary to Jensen's inequality, we find a local bound M to E

$$M = \max_{v \in B(u,\hat{\epsilon})} E(v) \leq \max_{v \in \Delta} E(v) \leq \max_{1 \leq i \leq n} E(u \pm \epsilon e_i)$$

□

Theorem 18. Continuity of Convex Functions *If $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex, then E is locally Lipschitz and hence continuous on int$(dom(E))$.*

Let us first define Lipschitz continuity.

Definition A function $E : \mathbb{R}^n \to \mathbb{R}^m$ is **Lipschitz continuous** with *Lipschitz constant* L if for all $u,\ v \in \mathrm{dom}(E)$

$$\|E(u) - E(v)\|_2 \leq L\|u - v\|_2$$

A function is **locally Lipschitz continuous** if for every $u \in \mathrm{dom}(E)$ there exists $\epsilon > 0$ such that $f_{|B(\epsilon,u)}$ is Lipschitz continuous

Proof. Let $B(u_0, \epsilon) \subset \mathrm{dom}(E)$ and $M = \sup_{u \in B(u_0,\epsilon)} E(u) < \infty$.
Consider $v \in B(u_0, \epsilon)$, $v \neq u_0$ and define

$$\alpha = \frac{1}{\epsilon}\|v - u_0\| \qquad\qquad z = u_0 + \frac{1}{\alpha}(v - u_0)$$

It is clear that $\|z - u_0\| = \epsilon$, $\alpha \leq 1$, and $v = \alpha z + (1 - \alpha)u_0$. By convexity of E then

$$E(v) \leq \alpha E(z) + (1 - \alpha)E(u_0) \leq E(u_0) + \alpha(M - E(u_0)) = E(u_0) + \frac{M - E(u_0)}{\epsilon}\|v - u_0\|$$

Now define $y = u_0 + \frac{1}{\alpha}(u_0 - v)$ with $\|y - u_0\| = \epsilon$ and $v = u_0 + \alpha(u_0 - y)$. We have

$$E(v) \geq E(u_0) + \alpha(E(u_0) - E(y)) \geq E(u_0) - \alpha(M - E(u_0)) = E(u_0) - \frac{M - E(u_0)}{\epsilon}\|v - u_0\|$$

As a result of the 2 inequalities

$$|E(v) - E(u)| \leq \frac{M - E(u_0)}{\epsilon}\|v - u_0\|.$$

□

2.3.3 Examples of Convex Functions

The next statements significantly increases our possibilities of constructing convex functions.

Lemma 19. *Given a closed convex function ϕ and a linear operator $\mathscr{A} : \mathbb{R}^m \to \mathbb{R}^n$, then $E(u) = \phi(\mathscr{A}(u))$ is closed and convex with*

$$dom(E) = \{u \in \mathbb{R}^m \quad : \mathscr{A}(u) \in dom(\phi)\}.$$

Proof. Let $\mathscr{A}(u) = Au + b = x \in \mathrm{dom}(\phi)$ and $\mathscr{A}(v) = Av + b = y \in \mathrm{dom}(\phi)$, then by convexity of ϕ for any $\theta \in [0,1]$ we have $\theta x + (1-\theta)y \in \mathrm{dom}(\phi)$ and

$$E[\theta u+(1-\theta)v] = \phi[\mathscr{A}(\theta u+(1-\theta)v)] = \phi[\theta(Au+b)+(1-\theta)(Av+b)] \leq \theta\phi(Au+b)+(1-\theta)\phi(Av+b) = \theta E(u)+(1-\theta)E(v).$$

This proves convexity of E. The closedness of its epigraph follows from continuity of the linear operator $\mathscr{A}$. □

Lemma 20. *Given two convex function E_1, E_2 and $\alpha_1, \alpha_2 > 0$, then $E = \alpha_1 E_1 + \alpha_2 E_2$ is convex with $dom(E) = dom(E_1) \cap dom E_2$.*

Proof. Let $u, v \in \mathrm{dom}(E_1) \cap \mathrm{dom}E_2$ and $\theta \in [0,1]$, by convexity of each E_1, E_2 we have

$$\begin{aligned} \alpha_1 E_1(\theta u + (1-\theta)v) + \alpha_2 E_2(\theta u + (1-\theta)v) &\leq \alpha_1 \theta E_1(u) + \alpha_1(1-\theta)E(v) + \alpha_2\theta E_1(u) + \alpha_2(1-\theta)E(v) \\ &= \theta[\alpha_1 E_1(u) + \alpha_2 E_2(u)] + (1-\theta)[\alpha_1 E_1(v) + \alpha_2 E_2(v). \end{aligned} \tag{2.4}$$

This proves the convexity of E. □

Taking into account that the following 1-dimensional functions are convex:

$$\begin{aligned} E(u) &= \exp(u) \\ E(u) &= |u|^p \quad p > 1 \\ E(u) &= |x| - \log(1 + |x|) \end{aligned}$$

the previous lemmas imply that the following multi-dimensional functions are convex:

$$\begin{aligned} E(u) &= \sum_{i=1}^{n} \exp(\alpha + \langle u, a_i \rangle) \\ E(u) &= |\langle u, a_i \rangle - b_i|^p \quad p > 1 \end{aligned}$$

Lemma 21. *Given two closed and convex function E_1, E_2, then $E(u) = \max\{E_1(u), E_1(u)\}$ is closed and convex with $dom(E) = dom(E_1) \cup dom(E_2)$.*

Proof. The epigraph is closed and convex because it is the intersection of two closed convex sets

$$\mathrm{epi}(E) = \{(u,t) \ : \ u \in \mathrm{dom}(E_1) \cap \mathrm{dom}(E_2),\ E_1(u) \leq t,\ E_2(u) \leq t\} = \mathrm{epi}(E_1) \cap \mathrm{epi}(E_2).$$

□

We have an even more general result.

Theorem 22. *Let D be some set, not necessarily convex or finite dimensional, and*

$$E(u) = \sup_{y \in C} \ \phi(u,y) \qquad \phi \textit{ closed and convex in } u \ \forall y \in D,$$

then E is closed and convex with $dom(E) = \{u \in \cap_{y\in D} dom(\phi(\cdot,y)) \ : \ \exists \gamma \in \mathbb{R} \ s.t. \ \phi(u,y) \leq \gamma \ \forall y \in D\}$.

Proof. We first show the definition of the domain. If u belongs to $\{u \in \cap_{y\in D}\mathrm{dom}(\phi(\cdot,y)) \ : \ \exists\gamma \in \mathbb{R} \text{ s.t. } \phi(u,y) \leq \gamma \ \ \forall y \in D\}$, then $E(u) < \infty$ and $u \in \mathrm{dom}(E)$. If u does not belong to this set, then there exists a sequence $\{y_k\}$ such that $\phi(u, y_k) \to \infty$ and u does not belong to $\mathrm{dom}(E)$.
$(u,t) \in \mathrm{epi}(E)$ if and only if for all $y \in D$, we have $u \in \mathrm{dom}(\phi(\cdot,y))$ and $\phi(u,y) \leq t$. As a results $\mathrm{epi}(E) = \cap_{y\in D}\mathrm{epi}(\phi(\cdot,y))$ is closed and convex as the intersection of closed and convex sets. □

As a result of this lemma, the function $E^*(y) = \sup_{u\in\mathrm{dom}(E)} \langle u, y\rangle - E(u)$ is convex for any E.

2.4 Existence and Uniqueness of Minimizers

It only makes sense to try to solve an optimization problem if it has a solution. Specially, if the solution is the limit of a relaxation sequence that is computed through costly iterative algorithm that might never converge. To show that a convex problem has a minimizer, we will see that it satisfies the necessary conditions to frame the problem in the more general framework of lower semi-continuous functions. This section explains the tools that we will use from this framework.

Definition Lower semi-continuity. A function $E : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous (l.s.c.), if for all u it holds that

$$\liminf_{v \to u} E(v) \geq E(u).$$

Theorem 23. *Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be l.s.c. and let there exist an α such that the sublevelset*

$$\{u \in \mathbb{R}^n \mid E(u) \leq \alpha\}$$

is nonempty and bounded, then there exists

$$\hat{u} \in \arg\min_u E(u).$$

Proof. Remember that the infimum is the largest lower bound on all possible values of $E(u)$ and consider a sequence $(u_k)_k$ such that $E(u_k) \to \inf_u E(u)$.
We distinguish two cases: For $\alpha = \inf_u E(u)$ the non-emptyness of S_α yields the assertion. For $\alpha > \inf_u E(u)$ it holds that from some sufficiently large k_0 on, we will have $u_k \in S_\alpha$. Since S_α is bounded there exists a convergent subsequence $u_{k_l} \to \bar{u}$. Due to the lower semi-continuity we find

$$\inf_u E(u) = \lim_{k \to \infty} E(u_k) = \lim_{l \to \infty} E(u_{k_l}) \geq E(\bar{u}).$$

Since by definition $\inf_u E(u) \leq E(\bar{u})$ we obtain equality and hence there exists $\bar{u} \in \operatorname{argmin}_u E(u)$. □

Theorem 24. Equivalence of l.s.c. and closedness. *For $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ the following two statements are equivalent*

- *E is lower semi-continuous (l.s.c.).*
- *E is closed (its epigraph is closed).*

Proof. Let E be closed and assume that E is not l.s.c. Then there exists a point u^0 and a sequence $(u_k)_k$ with $\lim_k u_k = u^0$ such that

$$\liminf_k E(u_k) < E(u^0).$$

In particular, there exists $\alpha \in \mathbb{R}$ and a subsequence $(u_{k_l})_{k_l}$ such that

$$E(u_{k_l}) \leq \alpha < E(u^0) \quad \forall k \tag{2.5}$$

Obviously, $(u_{k_l}, \alpha) \in \operatorname{epi}(E)$ for all k_l and $(u_{k_l}, \alpha) \to (u^0, \alpha)$, but according to (**??**) $(u^0, \alpha) \notin \operatorname{epi}(E)$, which contradicts the closedness of E.
To prove the other direction of the claim, let E be l.s.c. and assume that E is not closed. Then there exists a sequence $(u_k, \alpha_k) \in \operatorname{epi}(E)$ with $(u_k, \alpha_k) \to (u^0, \alpha^0) \notin \operatorname{epi}(E)$. We find

$$\liminf_k E(u_k) \leq \lim_k \alpha_k = \alpha^0 < E(u^0).$$

On the other hand, due to E being l.s.c. we have $E(u^0) \leq \liminf_k E(u_k)$, which is a contradiction. □

2.4.1 Existence of Minimizers of Convex Functions

Definition Coercivity. A function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called coercive if $E(v_n) \to \infty$ for all sequences $(v_n)_n$ with $\|v_n\| \to \infty$.

It is easy to proof that coercivity implies existence of a bounded sublevelset by contradiction. We have now all the tools to prove existence of minimizers of convex functions.

Theorem 25. Existence of a Minimizer *Let $E : \mathbb{R}^n \to \mathbb{R}$ be convex and coercive, then an element $\hat{u} \in \arg\min_u E(u)$ exists.*

Proof. As $\mathrm{dom}(E) = \mathbb{R}^n$ and E convex, E is Lipschitz continuous, and thus continuous. At the same time, as E is coercive, there exists a non-empty bounded sublevelset, and we can apply the theorem on the existence of minimizers for lower semi-continuous functions to prove existence of a minimizer. □

Theorem 26. Uniqueness. *If $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ is strictly convex, then there exists at most one local minimum which is the unique global minimum.*

Proof. Assume there are 2 global minima u, v with $u \neq v$, $E(u) = E(v)$, then any $\theta \in [0, 1]$ we have

$$E(\theta u + (1 - \theta)v) < \theta E(u) + (1 - \theta)E(v),$$

which contradicts the definition of $u,\ v$ as global minima. □

2.5 Subdifferentials

2.5.1 Supporting Hyperplanes

Up to now we were describing properties of convex functions in terms of function values or their gradients. When the function is not differentiable, we need to define a direction that acts as the gradient of differentiable functions and points onto the direction of maximum ascent. In convex analysis such directions are defined by supporting hyperplanes.

Definition Let C be a convex set. We say that hyperplane

$$\mathscr{H}(g, \gamma) = \{u \in \mathbb{R}^n \quad : \quad \langle g, u\rangle = \gamma, \quad g \neq 0\}$$

is supporting to C if any $u \in C$ satisfies $\langle g, u\rangle \leq \gamma$.
We say that the hyperplane $\mathscr{H}(g, \gamma)$ separates a point u_0 from C if

$$\langle g, u\rangle \leq \gamma \leq \langle g, u_0\rangle \ \ \forall u \in C.$$

Now we can enunciate two separation theorems necessary to define gradient-like directions for non-differentiable functions.

Theorem 27. Separating Hyperplane Theorem *Let C be a closed convex set and $u_0 \notin C$. Then there exists a hyperplane $\mathscr{H}(g, \gamma)$ that strictly separates u_0 from C.*

Proof. See Boyd and Vandenberghe, *Convex Optimization Theory*, pp 46–49. □

The next separation theorem deals with boundary points of convex sets.

Theorem 28. Supporting Hyperplane Theorem *Let C be a closed convex set and u_0 in the boundary of C. Then there exists a hyperplane $\mathscr{H}(g, \gamma)$ supporting to C and passing through u_0.*

Proof. See Boyd and Vandenberghe, *Convex Optimization Theory*, pp 50–51. □

2.5.2 The Subdifferential

We now have all the tools to introduce the notion of subdifferential that extends the notion of gradient to non-differentiable functions.

Definition Subdifferential. Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex, the subdifferential of E at u is

$$\partial E(u) = \{p \in \mathbb{R}^n \mid E(v) - E(u) - \langle p, v-u\rangle \geq 0,\ \forall v \in \mathbb{R}^n\}$$

- Elements of $\partial E(u)$ are called subgradients.
- If $\partial E(u) \neq \emptyset$, we call E subdifferentiable at u.
- By convention, $\partial E(u) = \emptyset$ for $u \notin \mathrm{dom}(E)$.

The subdifferential ∂E is necessary because subgradients are not unique. Consider for example a function as friendly-looking as the absolute value at zero:

$$\forall g \in [-1,1], \quad E(u) = |u| \geq gu = E(0) + \langle g, u-0\rangle$$

As a result, the subdifferential at 0 contains the interval $\partial E(0) = [-1,1]$. In general $\partial E(u)$ is a set. Form its definition as a set of linear constraints, we can easily see that it is closed and convex, in this case the interval $[-1,1]$.

2.5.3 Subdifferentiablity and Convexity

The subdifferentiability of a function is important because it implies its convexity.

Theorem 29. *If for any $u \in dom(E)$ the subdifferential $\partial E(u)$ is non-empty, then E is a convex function.*

Proof. Given $u, v \in \mathrm{dom}(E)$, and $\theta \in [0,1]$, let $u_\theta = \theta u + (1-\theta)v$. As the subdifferential $\partial E(u_\theta)$ is non-empty, we can pick $g \in \partial E(u_\theta)$ satisfying

$$\begin{aligned} E(u_\theta) &\geq E(v) + \langle g, v - u_\theta\rangle = E(v) + \theta\langle g, v-u\rangle \\ E(u_\theta) &\geq E(u) + \langle g, u - u_\theta\rangle = E(u) - (1-\theta)\langle g, v-u\rangle. \end{aligned}$$

Multiplying the first inequality by $1-\theta$, the second by θ, and adding the results, we get the inequality that defines a convex function $E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$. □

The converse statement is also true.

Theorem 30. *If E is a closed convex function and $u \in int(dom(E))$, then $\partial E(u)$ is a non-empty bounded set.*

Proof. Note that the point $(E(u), u)$ belongs to the boundary of $\mathrm{epi}(E)$, which is convex. As a result, there exists a hyperplane $\mathscr{H} = (g, \gamma)$ supporting to $\mathrm{epi}(E)$ at $(E(u), u)$:

$$\gamma\tau + \langle g, u\rangle \leq \gamma E(u) + \langle g, u\rangle \quad \forall (u,\tau) \in \mathrm{epi}(E)$$

Without loss of generality, we can assume $\|g\|^2 + \gamma^2 = 1$. We can determine the sign of γ by checking the inequality for any point in the epigraph. In particular for any $\tau \geq E(u)$, we have $(u,\tau) \in \mathrm{epi}(E)$ that results in $\gamma > 0$.

To find a subgradient $p \in \partial E(u)$, we will use that a convex function is locally upper bounded in the interior of its domain. That is, there is some $\epsilon > 0, M > 0$ such that $B(u,\epsilon) \subset \mathrm{dom}(E)$ and

$$E(v) - E(u) \leq M\|v - u\| \quad \forall v \in B(u,\epsilon)$$

For any v from this ball, the supporting hyperplane equation reads

$$\langle g, v-u\rangle \leq \gamma(E(v)-E(u)) \leq \gamma M\|v-u\|$$

In particular, if we choose $v = u + \epsilon g$ we get $\|g\|^2 \leq M\gamma\|d\|$. Plugging now the condition $\|g\|^2 + \gamma^2 = 1$ we get

$$\gamma \geq \frac{1}{\sqrt{1+M^2}}.$$

If we choose $p = \frac{g}{\gamma}$ we obtain

$$E(v) \geq E(u) + \langle p, v-u\rangle \quad \forall v \in \mathrm{dom}(E)$$

and p is a subgradient of E at u. Finally, to show that the subdifferential is bounded we assume that $p \neq 0$ and consider the point $v = u + \epsilon\frac{p}{\|p\|}$ such that

$$\epsilon\|p\| = \langle p, v-u\rangle \leq E(v)-E(u) \leq M\|v-u\| = M\epsilon$$

Thus, $\partial E(u)$ is bounded by M. □

The conditions of this theorem cannot be relaxed. For instance, the function $E(u) = -\sqrt{u}$ is convex and closed in its domain $\{u\colon u \geq 0\}$, but its subdifferential does not exists at the only point (0) that is not in its interior. This is just another reminder that considering the interior of the domain for convex functions is important.

To conclude this section, let us point out to the property of the subgradients that makes it important for optimization.

Theorem 31. Optimality Condition. $0 \in \partial E(\hat{u})$ *if and only if* $\hat{u} \in \arg\min_{u\in\mathbb{R}^n} E(u)$.

Proof. If $0 \in \partial E(\hat{u})$, by definition of the subgradient

$$E(u) \geq E(\hat{u}) + \langle 0, u-\hat{u}\rangle = E(\hat{u}) \quad \forall u \in \mathrm{dom}(E)$$

and we conclude that $\hat{u}$ is a minimizer of E. On the other hand, if $E(u) \geq E(\hat{u})$ for all $u \in \mathrm{dom}(E)$, then 0 satisfies the condition of subgradient of E at $\hat{u}$. □

2.5.4 Alternative Definitions of Subgradients

The supporting hyperplane theorem appears on the proof of the "subdifferentiability" theorem because subgradients can be interpreted in terms of supporting hyperplanes.

Theorem 32. Geometric interpretation of Subgradients. *Any subgradient* $p \in \partial E(u)$ *represents a non-vertical supporting hyperplane to epi*(E) *at* $(u, E(u))$.

Proof. Let $p \in \partial E(u)$. Then, by definition of subgradient,

$$\begin{aligned}
&E(v)-E(u)-\langle p, v-u\rangle \geq 0 && \forall v \in \mathbb{R}^n\\
&\alpha - E(u) - \langle p, v-u\rangle \geq 0 && \forall (v,\alpha) \in \mathrm{epi}(E)\\
&\left\langle \begin{bmatrix}-p\\1\end{bmatrix}, \begin{bmatrix}v\\\alpha\end{bmatrix} - \begin{bmatrix}u\\E(u)\end{bmatrix}\right\rangle \geq 0 && \forall (v,\alpha) \in \mathrm{epi}(E).
\end{aligned}$$

As a result, the non-vertical hyperplane $\mathscr{H} = (g,\gamma)$ with $g = (-p, 1)$ and $\gamma = \langle p, u\rangle - E(u)$ supports epi(E) at $(u, E(u))$. □

Apart from this geometric interpretation, it is useful to compute the subdifferential of a differentiable function to understand why it is a generalization of the gradient. The next theorem does that.

Theorem 33. Subdifferential of of Differentiable Functions. *Let the convex function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be differentiable at $u \in int(dom(E))$. Then*

$$\partial E(u) = \{\nabla E(u)\}.$$

Proof. The subdifferential $\partial E(u)$ of some convex E at $u \in \mathrm{dom}(f)$ is given as

$$\{p \in \mathbb{R}^n : E(z) - E(u) - \langle p, z - u\rangle \geq 0, \forall\, z \in \mathrm{dom}(f)\}\,.$$

Since $u \in \mathrm{int}(\mathrm{dom}(E))$, we find that for all $v \in \mathbb{R}^n$, $z = u \pm \epsilon v \in \mathrm{dom}(E)$ for ϵ small enough. Therefore, it holds that

$$E(u + \epsilon v) \geq E(u) + \epsilon\langle p, v\rangle, \quad E(u - \epsilon v) \geq E(u) - \epsilon\langle p, v\rangle,$$

for all $v \in \mathbb{R}^n$ and ϵ small enough. This implies that

$$\lim_{\epsilon\to 0} \frac{E(u + \epsilon v) - E(u)}{\epsilon} \geq \langle p, v\rangle, \quad \lim_{\epsilon\to 0} \frac{E(u) - E(u - \epsilon v)}{\epsilon} \leq \langle p, v\rangle,$$

which means

$$\langle \nabla E(u), v\rangle \geq \langle p, v\rangle, \quad \langle \nabla E(u), v\rangle \leq \langle p, v\rangle,$$

i.e.

$$\langle \nabla E(u) - p, v\rangle = 0$$

for all $v \in \mathbb{R}^n$. For the particular choice of $v := \nabla E(u) - p$ we find $p = \nabla f(u)$. The above concludes the proof if we can show that $\partial f(u)$ is non-empty, which follows from the Theorem on Subdifferentiability. □

2.5.5 Subdifferential Rules

In the same way that the gradient of a differentiable function is only defined for points in the interior of the domain, the subdifferential of a proper convex function is always defined for points in the relative interior of its domain.

The relative interior of a set is a refinement of the concept of the interior that is useful when dealing with low-dimensional sets embedded in higher-dimensional spaces. Intuitively, the relative interior of a set contains all points that are not on the "edge" of the set, relative to the smallest subspace in which this set lies. When the set is convex, the definition takes the following simple form:

Definition Relative Interior of Convex Sets The relative interior of a convex set C is defined as

$$\mathrm{ri}(C) := \{x \in C \mid \forall y \in C,\ \exists \lambda > 1,\ \text{s.t. } \lambda x + (1 - \lambda)y \in C\}$$

As mentioned earlier, the subdifferentiability of convex functions can be guaranteed for points that are not necessarily in the interior of the domain, but that are in its relative interior. To better understand this difference, consider the line segment $I = [-1, 1]$ as a convex subset of the Euclidean plane $I \subset \mathbb{R}^2$. The interior of I is empty with the Euclidean topology of $\mathbb{R}^2$, but its relative interior is the open line segment $\mathrm{ri}(I) = (0, 1)$.

One key property of the relative interior is that it is not empty for convex sets.

Theorem 34. *Let C be a non-empty convex set, then $ri(C)$ is not empty.*

Now that we understand where subdifferentials exists, we can learn the rules that guide their computation.

Theorem 35. Sum Rule. *Let E_1, E_2 be convex functions, then $\partial(E_1 + E_2)(u) = \partial E_1(u) + \partial E_2(u)$ for all $u \in ri(dom(E_1)) \cap ri(dom(E_2))$.*

Proof. See Nesterov, *Introductory Lectures on Convex Optimization*, Lemma. 3.1.9. $\square$

Theorem 36. Chain Rule *Given the linear operator $A \in \mathbb{R}^{m\times n}$ and the convex function $E : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$, then $\partial(E \circ A)(u) = A^* \partial E(Au)$ for all $u \in ri(dom(E)) \cap range(A)$.*

Proof. See Nesterov, *Introductory Lectures on Convex Optimization*, Nesterov, Lemma. 3.1.8. $\square$

Chapter 3

Fixed-Point Iterations

Convex optimization problems come in so many shapes and sizes that the algorithms developed to solve them form a zoo. Each algorithm exploits a particular feature of the convexity of the objective function or the constrain set to find the solution of the problem. As a result, we traditionally also analyze the convergence of each algorithm and its properties in a case by case manner.

It is possible to interpret many of these algorithms as fixed-point iterations in a unified manner and analyze their convergence with the same approach. To do so, we first need to formulate the optimization problem as finding a zero of a monotone operator. This problem is converted into the problem of finding a fixed point of a function and solved by the fixed point iteration algorithm. Different choices of the monotone operator and fixed point function result in different well-known algorithms.

This new view on many classic algorithms provides a convenient strategy to analyze their convergence with a single approach. The price to pay, however, is an additional level of abstraction that might at first seem disconnected from intuitive algorithms like gradient descent. Be patient, and read on.

The material of this chapter is taken mostly from: Ryu and Boyd, *Primer on Monotone Operator Methods*, 2016.

3.1 Nonexpansive mappings and contractions

Definition A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is a **contraction** if it is Lipschitz continuous with constant $L < 1$, that is,

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x,\ y \in \operatorname{dom}(F).$$

When $L = 1$, we say that F is a **nonexpansive** operator.

In other words, mapping a pair of points by a contraction reduces the distance between them; mapping them by a nonexpansive operator does not increase the distance between them. See Figure **??**.

Theorem 37. *If F is nonexpansive and $dom(F) = \mathbb{R}^n$, then its set of fixed points*

$$\{x \in dom(F)\ :\ x = F(x)\}$$

is closed and convex. If F is a contraction and $dom(F) = \mathbb{R}^n$, its has exactly one fixed point.

Proof. Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be nonexpansive and denote by X the set of its fixed points. Note that we can also define $X = (I - F)^{-1}(\{0\})$, where I is the identity function. From this definition, X is closed because it is the preimage of a continuous function $(F - I)$ on a closed set $(\{0\})$. To show that it is convex, let

(a) Contraction Mapping (b) Nonexpansive Mapping

Fig. 3.1: Illustration of a contractive and a nonexpansive mapping F on two points. Source: Ryu and Boyd, *Primer on Monotone Operator Methods*, 2016

$x, y \in \operatorname{dom}(F)$ and $\theta \in [0,1]$ and define $z = \theta x + (1-\theta)y$. We will show that $z \in X$. As F is nonexpansive

$$
\begin{aligned}
\|Fz - x\| &= \|Fz - Fx\| \le \|z - x\| = (1-\theta)\|x - y\| \\
\|Fz - y\| &= \|Fz - Fy\| \le \|z - y\| = \theta\|x - y\| \\
\|x - y\| &\le \|Fz - x\| + \|Fz - y\| \le \|x - y\|
\end{aligned}
$$

From the last inequality we know that Fz is on the line segment between x and y. In particular, as $\|Fz - y\| = \theta\|x - y\|$, we have $Fz = \theta x + (1-\theta)y = z$ and z is a fixed point of F.

Let $x,\ y \in X$ be again fixed points of F and F be now a contraction with Lipschitz constant $L < 1$, then

$$\|x - y\| = \|Fx - Fy\| \le L\|x - y\|.$$

This is a contradiction unless $x = y$, which implies that there is a single fixed-point. □

Lemma 38. *Convex combinations as well as compositions of nonexpansive operators are nonexpansive.*

Proof. If $F_1 : \mathbb{R}^n \to \mathbb{R}^n$ has Lipschitz constant L_1 and $F_2 : \mathbb{R}^n \to \mathbb{R}^n$ has Lipschitz constant L_2, then the composition F_2F_1 has Lipschitz constant L_2L_1. Indeed, let $x,\ y \in \operatorname{dom}(F_1)$ such that $F_1x,\ F_1y \in \operatorname{dom}(F_2)$ then

$$\|F_2F_1x - F2F1y\| \le L_2\|F_1x - F1y\| \le L_2L_1\|x - y\|$$

As a result, the composition of nonexpansive operators is nonexpansive, and the composition of a contraction and a nonexpansive operator is a contraction.

Similarly, if $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1F_1 + \alpha_2F_2$ has Lipschitz constant $|\alpha_1|L_1 + |\alpha_2|L_2$. Indeed, let $x,\ y \in \operatorname{dom}(F_1) \cap \operatorname{dom}(F_2)$, then

$$
\begin{aligned}
\|(\alpha_1F_1 + \alpha_2F_2)x - (\alpha_1F_1 + \alpha_2F_2)y\| &\le \|\alpha_1F_1x - \alpha_1F_1y\| + \|\alpha_2F_2x - \alpha_2F_2y\| \\
&\le |\alpha_1|L_1\|x - y\| + |\alpha_2|L_2\|x - y\| \\
&\le (|\alpha_1|L_1 + |\alpha_2|L_2)\|x - y\|. \qquad (3.1)
\end{aligned}
$$

As a result, a weighted average of nonexpansive operators $\theta F_1 + (1-\theta)F_2$ with $\theta \in [0,1]$ is also nonexpansive. If one of them is a contraction and $\theta \in (0,1)$, then the weighted average is a contraction. □

There are very few examples of contractions that are useful in the convex optimization for vision and learning applications. Most of the time we work with nonexpansive operators. Among them, a particular type called averaged operator, is specially useful and common.

Definition An operator G is **averaged** if $G = (1-\alpha)I + \alpha R$ for some $\alpha \in (0,1)$ and nonexpansive R.

G is nonexpansive because it is a convex combination of nonexpansive operators (the identity I is non-expansive). Moreover, it is easy to see that G has the same fixed points as R.

$$u^* = Ru^* \;\Leftrightarrow\; (1-\alpha)u^* + \alpha u^* = (1-\alpha)u^* + \alpha Ru^* \;\Leftrightarrow\; u^* = [(1-\alpha)I + \alpha R]\, u^* = Gu^* \tag{3.2}$$

We will use this property to design algorithms that find fixed points of nonexpansive operators R and are parametrized by $\alpha \in (0,1)$.

3.1.1 Properties of Averaged Operators

Lemma 39. *If a function $G : \mathbb{R}^n \to \mathbb{R}^n$ is averaged with respect to $\alpha \in (0,1)$, then it is also averaged with respect to any other parameter $\tilde{\alpha} \in (0,\alpha)$.*

Proof. Since G is averaged with respect to α there exists a nonexpansive operator R such that $G = \alpha I + (1-\alpha)R$. We find

$$\begin{aligned} G &= \alpha I + (1-\alpha)R \\ &= \tilde{\alpha} I + (\alpha - \tilde{\alpha})I + (1-\alpha)R \\ &= \tilde{\alpha} I + (1-\tilde{\alpha}) \underbrace{\left(\frac{\alpha - \tilde{\alpha}}{1-\tilde{\alpha}} I + \frac{1-\alpha}{1-\tilde{\alpha}} R \right)}_{=:\tilde{R}}. \end{aligned}$$

And $\tilde{R}$ is still nonexpansive because

$$\begin{aligned} \|\tilde{R}(u) - \tilde{R}(v)\| &\le \frac{\alpha - \tilde{\alpha}}{1-\tilde{\alpha}} \|u - v\| + \frac{1-\alpha}{1-\tilde{\alpha}} \|R(u) - R(v)\| \\ &\le \frac{\alpha - \tilde{\alpha}}{1-\tilde{\alpha}} \|u - v\| + \frac{1-\alpha}{1-\tilde{\alpha}} \|u - v\| \\ &= \|u - v\|. \end{aligned}$$

□

Lemma 40. *If $G_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $G_2 : \mathbb{R}^n \to \mathbb{R}^n$ are averaged, then $G_2 \circ G_1$ is also averaged.*

Proof. Let $G_1 = \alpha_1 I + (1-\alpha_1)R_1$ and $G_2 = \alpha_2 I + (1-\alpha_2)R_2$ for nonexpansive operators R_1 and R_2. Then

$$\begin{aligned} G_2(G_1)(u) &= \alpha_2 G_1(u) + (1-\alpha_2)R_2(G_1(u)) \\ &= \alpha_1\alpha_2 u + \alpha_2(1-\alpha_1)R_1(u) + (1-\alpha_2)R_2(G_1(u)) \\ &= \alpha_1\alpha_2 u + (1-\alpha_1\alpha_2)\left(\frac{\alpha_2(1-\alpha_1)}{1-\alpha_1\alpha_2} R_1(u) + \frac{(1-\alpha_2)}{1-\alpha_1\alpha_2} R_2(G_1(u)) \right). \end{aligned}$$

Since the concatenation of nonexpansive operators is nonexpansive, and convex combinations of nonexpansive operators are nonexpansive, we conclude that $G_2 \circ G_1$ is averaged. □

It is possible to determine if an operator is averaged without explicitly finding its decomposition into a convex combination of the identity and a nonexpansive operator. We do so through the notion of firmly nonexpansive operators.

Definition A function $G : \mathbb{R}^n \to \mathbb{R}^n$ is called **firmly nonexpansive**, if for all $u, v \in \mathbb{R}^n$ it holds that

$$\|G(u) - G(v)\|_2^2 \le \langle G(u) - G(v), u - v \rangle.$$

Lemma 41. *A function $G : \mathbb{R}^n \to \mathbb{R}^n$ is firmly nonexpansive if and only if G is averaged with $\alpha = \frac{1}{2}$.*

Proof. First, let G be averaged with $\alpha = 1/2$, i.e., $G = \frac{1}{2}I + \frac{1}{2}R$ for some nonexpansive operator $R = 2G - I$. As R is nonexpansive, we have

$$\begin{aligned}\|u-v\|_2^2 \geq \|R(u)-R(v)\|^2 &= \|2G(u)-2G(v)-(u-v)\|^2 \\ &= 4\|G(u)-G(v)\|^2 - 4\langle G(u)-G(v), u-v\rangle + \|u-v\|^2,\end{aligned}$$

which implies $\langle G(u)-G(v), u-v\rangle \geq \|G(u)-G(v)\|_2^2$ and shows that G is firmly nonexpansive.

Second, let G be firmly nonexpansive and define $R = 2G - I$, then

$$\begin{aligned}\|R(u)-R(v)\|^2 &= \|2G(u)-2G(v)-(u-v)\|^2 \\ &= 4\|G(u)-G(v)\|^2 - 4\langle G(u)-G(v), u-v\rangle + \|u-v\|^2 \\ &\leq \|u-v\|^2,\end{aligned}$$

which shows that R is nonexpansive, i.e., $G = \frac{1}{2}I + \frac{1}{2}R$ is averaged with $\alpha = 1/2$. □

3.2 Fixed-point Iterations

We are now ready to discuss the main algorithm of this chapter.

Definition Let $G\colon \mathbb{R}^n \to \mathbb{R}$, and $u^0 \in \mathbb{R}^n$ be a starting point, the **fixed-point or Picard iteration** is

$$u^{k+1} = G(u^k).$$

As the name suggests, the fixed-point iteration is used to find a fixed point u of G. Using this iteration to solve an optimization problem involves two steps: 1) find a suitable G whose fixed points are solutions to the problem at hand, 2) show that the iteration converges to a fixed point. For this second step, we show two simple conditions that guarantee convergence.

Theorem 42. Banach fixed-point theorem. *If the update rule $G : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction with Lipschitz constant $L < 1$, then the fixed-point iteration converges to the unique fixed-point $\hat{u}$ of G with*

$$\|u^k - \hat{u}\| \leq L^k \|u^0 - \hat{u}\|.$$

Proof. See Ryu and Boyd, *Primer on Monotone Operator Methods*, 2016. □

Theorem 43. Krasnosel'skii-Mann Theorem. *If the operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is averaged and has a fixed-point, then the iteration*

$$u^{k+1} = G(u^k)$$

converges to a fixed point of G for any starting point $u^0 \in \mathbb{R}^n$.

Proof. We'll make use of the identity

$$\|(1-\theta)a + \theta b\|^2 = (1-\theta)\|a\|^2 + \theta\|b\|^2 - \theta(1-\theta)\|a-b\|^2,$$

which holds for any $\theta \in \mathbb{R}$, $a, b \in \mathbb{R}^n$. It can be verified by expanding both sides as a quadratic function of θ. The first two terms correspond to the definition of convexity for function $\|\cdot\|^2$, the third one improves this bound.

Because G is averaged, there exists a non-expansive mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $G = (1-\theta)I + \theta T$. Recall that T has the same fixed points as F. We consider the fixed point iteration

$$u^{k+1} = G(u^k) = (1-\theta)u^k + \theta T u^k.$$

Denote by U the (nonempty) set of fixed-points of G and let $u^* \in U$, that is, $G(u^*) = u^*$. Then we have

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &= \|(1-\theta)(u^k - u^*) + \theta(Tu^k - u^*)\|^2 \\
&= (1-\theta)\|u^k - u^*\|^2 + \theta\|Tu^k - u^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2 \\
&= (1-\theta)\|u^k - u^*\|^2 + \theta\|Tu^k - Tu^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2 \qquad (*) \\
&\leq (1-\theta)\|u^k - u^*\|^2 + \theta\|u^k - u^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2 \\
&= \|u^k - u^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2
\end{aligned}$$

This shows that the distance to the solution set decreases at each step. We call this property Fejèr monotonicity.

Applying the inequality k times yields

$$\|u^{k+1} - u^*\|^2 \leq \|u^0 - u^*\|^2 - \theta(1-\theta)\sum_{j=0}^{k}\|Tu^j - u^j\|^2$$

and hence

$$\sum_{j=0}^{k}\|Tu^j - u^j\|^2 \leq \frac{\|u^0 - u^*\|^2 - \|u^{k+1} - u^*\|^2}{\theta(1-\theta)} \leq \frac{\|u^0 - u^*\|^2}{\theta(1-\theta)}.$$

As the upper bound does not depend on k, the series of non-negative terms remains bounded as $k \to \infty$ and we conclude that $\|Tu^k - u^k\| \to 0$ as $k \to \infty$.

From that we can also estimate a convergence rate of the fixed-point residual:

$$\min_{j=0\ldots k}\|Tu^j - u^j\|^2 \leq \frac{\|u^0 - u^*\|^2}{(k+1)\theta(1-\theta)},$$

Since the iterates $\{u^k\}_{k=1}^{\infty}$ lie in the compact set (due to the Fejèr monotonicity)

$$\{u^k\}_{k=1}^{\infty} \subset C = \left\{v \mid \|v - u^*\| \leq \|u^0 - u^*\|\right\},$$

there exists at least one subsequence $\{u^{k_l}\}_{l=1}^{\infty}$ which converges to some point $\widehat{u}$.

Since $Tu^{k_l} - u^{k_l} \to 0$, we also have that $Gu^{k_l} - u^{k_l} = (G-I)u^{k_l} \to 0$. Since $G-I$ is Lipschitz continuous (as T is nonexpansive) and hence continuous, we have that $G\widehat{u} = \widehat{u}$ and hence the subsequence converges to a point in $\widehat{u} \in U$.

As (**??**) holds for any point from $u^* \in U$, we can apply it the point $\widehat{u}$ our subsequence converges to. We know that for the iterates of the original sequence the distance to this point is monotonically decreasing,

$$\|u^{k+1} - \widehat{u}\| \leq \|u^k - \widehat{u}\|.$$

Since a subsequence $\{u^{k_l}\}_{l=1}^{\infty}$ of $\{u^k\}_{k=1}^{\infty}$ is converging to $\widehat{u}$, and $\|u^k - \widehat{u}\|$ is monotonically decreasing, we have convergence of the entire sequence to $\widehat{u}$. □

3.3 Gradient Descent as an Averaged Operator

Given a differentiable convex function $E\colon \mathbb{R}^n \to \mathbb{R}$, consider the problem

$$u \in \arg\min_{u \in \mathbb{R}^n} E(u).$$

The first-order optimality conditions of the problem characterize the solution $u^\star$ by

$$\nabla E(u^\star) = 0 \iff u^\star = (I - \tau\nabla E)u^\star$$

for any $\tau \neq 0$. The fixed-point iteration for this setup is

$$u^{k+1} = u^k - \tau \nabla E(u^k).$$

This algorithm is called **gradient descent** with a constant step size $\tau > 0$. To guarantee convergence of this fixed-point iteration, we need to determine under which conditions $(I - \tau \nabla E)$ is a contraction or an averaged operator. To this purpose, we will use the following result.

Theorem 44. Baillon-Haddad theorem. *A continuously differentiable convex function $E : \mathbb{R}^n \to \mathbb{R}$ is L-smooth if and only if $\frac{1}{L}\nabla E$ is firmly nonexpansive, i.e.*

$$\langle \nabla E(u) - \nabla E(v), u - v \rangle \geq \frac{1}{L} \|\nabla E(u) - \nabla E(v)\|_2^2$$

for all $u, v \in \mathbb{R}^n$.

Proof. See Nesterov, *Introductory Lectures on Convex Optimization*, Theorem 2.1.5. □

We can now determine the conditions under which gradient descent with a constant step size converges.

Theorem 45. *If $E : \mathbb{R}^n \to \mathbb{R}$ has a minimizer, is convex, and L-smooth, then the gradient descent iteration with constant step size $\tau \in (0, \frac{2}{L})$ converges to a minimizer.*

Proof. We will show that the fixed-point operator of gradient descent $G(u) = u - \tau \nabla E(u)$ is averaged.

By Baillon-Haddad theorem, we know that $\frac{1}{L}\nabla E$ is firmly non-expansive, or equivalently, averaged with $\alpha = 1/2$. Let $\frac{1}{L}\nabla E = \frac{1}{2}(I + T)$ for a non-expansive T, it holds

$$G(u) = u - \tau L \frac{1}{L} \nabla E(u) = \left(1 - \frac{L\tau}{2}\right) I + \frac{L\tau}{2}(-T)$$

It is clear that if T is non-expansive, $(-T)$ is also non-expansive, and consequently G is averaged for $\tau \in (0, \frac{2}{L})$. □

Theorem 46. *If $E : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with parameter m and strongly smooth with parameter L, then the gradient descent iteration with constant step size $\tau \in (0, \frac{2}{L})$ converges to the unique minimizer $u^\star$ with geometric convergence rate*

$$\|u^k - u^\star\| \leq c^k \|u^0 - u^\star\|.$$

Proof. We will show that $(I - \tau \nabla E)$ is Lipschitz with parameter $c = \max\{|1 - \tau m|, |1 - \tau L|\}$. To simplify the proof, we will assume that E is twice continuously differentiable although the result is still true without this assumption. If E is twice continuously differentiable, we have

- $D(I - \tau \nabla E) = I_n - \tau \nabla^2 E$, where I_n is the identity matrix
- m strong convexity is equivalent to $\nabla^2 E \succeq m I_n$
- L-smoothness corresponds to $\nabla^2 E \preceq L I_n$

Putting these together, we have

$$\begin{aligned} (1 - \tau L) I_n \preceq D(I - \tau \nabla E) \preceq (1 - \tau m) I_n \\ \|D(I - \tau \nabla E)\| \leq \max\{|1 - \tau m|, |1 - \tau L|\} \\ (I - \tau \nabla E) \text{ has Lipscitz constant } c = \max\{|1 - \tau m|, |1 - \tau L|\}. \end{aligned} \tag{3.3}$$

As a result, $(I - \tau \nabla E)$ is a contraction for $\tau \in (0, \frac{2}{L})$ and the fixed-point iteration converges to the unique fixed point of the contraction with the geometric rate c^k by Banach fixed-point theorem.

□