

Weekly Exercises 1

Room: 02.09.023

Friday, 03.11.2017, 09:15-11:00

Submission deadline: Monday, 30.10.2017, 10:15, Room 02.09.023

Theory: Convex Sets and Functions (12+8 Points)

Exercise 1 (4 Points). Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper. Prove the equivalence of the following statements:

- f is convex.
- $\text{epi}(f) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+1} : f(x) \leq y \right\}$ is convex.

Exercise 2 (4 Points). Let $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex. Show that the perspective function $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ of g given as

$$f(x, t) := \begin{cases} t g\left(\frac{x}{t}\right) & \text{if } t > 0 \text{ and } \frac{x}{t} \in \text{dom}(g) \\ +\infty & \text{otherwise,} \end{cases}$$

is convex.

Exercise 3 (4 Points). Let $\emptyset \neq X \subset \mathbb{R}^n$. Prove the equivalence of the following statements:

- X is closed.
- Every convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ attains its limit in X .

Exercise 4 (4 Points). Let $X \subset \mathbb{R}^n$ open and convex and let $f : X \rightarrow \mathbb{R}$ be twice continuously differentiable. Prove the equivalence of the following statements:

- f is convex.
- For all $x \in X$ the Hessian $\nabla^2 f(x)$ is positive semidefinite ($\forall v \in \mathbb{R}^n : v^\top \nabla^2 f(x) v \geq 0$).

Hints: You can use that for $x, y \in X$ it holds that f is convex iff

$$(y - x)^\top \nabla f(x) \leq f(y) - f(x).$$

Further recall that there are two variants of the Taylor expansion:

$$f(x + tv) = f(x) + tv^\top \nabla f(x) + \frac{t^2}{2} v^\top \nabla^2 f(x) v + o(t^2)$$

with $\lim_{t \rightarrow 0} \frac{o(t^2)}{t^2} = 0$ and

$$f(x + v) = f(x) + v^\top \nabla f(x) + \frac{1}{2} v^\top \nabla^2 f(x + tv) v$$

for appropriate $t \in (0, 1)$.

Exercise 5 (4 Points). Let $X \subset \mathbb{R}^n$ open and convex, $A \in \mathbb{R}^{n \times n}$ positive semidefinite, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$. Show that the quadratic form $f : X \rightarrow \mathbb{R}$ defined as

$$f(x) := \frac{1}{2} x^\top A x + b^\top x + c,$$

is convex.

Programming: Inpainting (12 Points)

Exercise 6 (12 Points). Write a MATLAB program that solves the inpainting problem for the vegetable image:

$$\min_{u \in \mathbb{R}^{n \times m}} \sum_{i,j} (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t.} \quad u_{i,j} = f_{i,j} \quad \forall (i,j) \in I,$$

with index set I of pixels to keep. Those can be identified as the white pixels of the mask image.

Hint: The constrained optimization problem can be reformulated so that it becomes unconstrained: Rewrite the objective as a least squares problem in terms of the unknown intensities $u_{i,j}$, $(i,j) \notin I$ using sparse linear operators: Find linear operators X, Y s.t. u can be decomposed as

$$u = X\tilde{u} + Yf$$

where \tilde{u} contains only the unknown intensities. Optimize for \tilde{u} instead of u . You may use MATLAB's `mldivide`.