Convex Optimization for Machine Learning and Computer Vision

Lecture: Dr. Virginia Estellers
Exercises: Emanuel Laude
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 1

Room: 02.09.023

Friday, 02.09.023, 09:15-11:00

Submission deadline: Monday, 30.10.2017, 10:15, Room 02.09.023

Theory: Convex Sets and Functions (12+8 Points)

Exercise 1 (4 Points)**.** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper. Prove the equivalence of the following statements:

- f is convex.
- $\operatorname{epi}(f) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+1} : f(x) \leq y \right\}$ is convex.

Exercise 2 (4 Points)**.** Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex. Show that the perspective function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ of g given as

$$f(x,t) := \begin{cases} t\, g\left(\frac{x}{t}\right) & \text{if } t > 0 \text{ and } \frac{x}{t} \in \operatorname{dom}(g) \\ +\infty & \text{otherwise,} \end{cases}$$

is convex.

Exercise 3 (4 Points)**.** Let $\emptyset \neq X \subset \mathbb{R}^n$. Prove the equivalence of the following statements:

- X is closed.
- Every convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ attains its limit in X.

Exercise 4 (4 Points)**.** Let $X \subset \mathbb{R}^n$ open and convex and let $f : X \to \mathbb{R}$ be twice continuously differentiable. Prove the equivalence of the following statements:

- f is convex.
- For all $x \in X$ the Hessian $\nabla^2 f(x)$ is positive semidefinite ($\forall\, v \in \mathbb{R}^n : v^\top \nabla^2 f(x) v \geq 0$).

Hints: You can use that for $x, y \in X$ it holds that f is convex iff

$$(y - x)^\top \nabla f(x) \leq f(y) - f(x).$$

Further recall that there are two variants of the Taylor expansion:

$$f(x + tv) = f(x) + tv^\top \nabla f(x) + \frac{t^2}{2} v^\top \nabla^2 f(x) v + o(t^2)$$

with $\lim_{t \to 0} \frac{o(t^2)}{t^2} = 0$ and

$$f(x + v) = f(x) + v^\top \nabla f(x) + \frac{1}{2} v^\top \nabla^2 f(x + tv) v$$

for appropriate $t \in (0, 1)$.

Exercise 5 (4 Points)**.** Let $X \subset \mathbb{R}^n$ open and convex, $A \in \mathbb{R}^{n \times n}$ positive semidefinite, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$. Show that that the quadratic form $f : X \to \mathbb{R}$ defined as

$$f(x) := \frac{1}{2} x^\top A x + b^\top x + c,$$

is convex.

Programming: Inpainting (12 Points)

Exercise 6 (12 Points)**.** Write a MATLAB program that solves the inpainting problem for the vegetable image:

$$\min_{u \in \mathbb{R}^{n \times m}} \sum_{i,j} (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t. } u_{i,j} = f_{i,j} \ \forall (i,j) \in I,$$

with index set I of pixels to keep. Those can be identified as the white pixels of the mask image.
Hint: The constrained optimization problem can be reformulated so that it becomes unconstrained: Rewrite the objective as a least squares problem in terms of the unknown intensities $u_{i,j}$, $(i,j) \notin I$ using sparse linear operators: Find linear operators X, Y s.t. u can be decomposed as

$$u = X\tilde{u} + Yf$$

where $\tilde{u}$ contains only the unknown intensities. Optimize for $\tilde{u}$ instead of u. You may use MATALBs `mldivide`.