Convex Optimization for Machine Learning and Computer Vision

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Winter Semester 2017/18

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# Weekly Exercises 5

Room: 02.09.023

Friday, 02.12.2017, 09:15-11:00

Submission deadline theory: Monday, 27.11.2017, 10:15, Room 02.09.023

Submission deadline coding: Monday, 04.12.2017, 10:15, Room 02.09.023

## Theory: Fenchel Duality (10+6 Points)

**Exercise 1** (4 Points)**.** Compute the convex conjugates of the following functions:

1. $f_1 : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ where $f_1(X) = \|X\|_{2,\infty}$.
2. $f_2 : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ where $f_2(X) = \delta_{\|\cdot\|_{2,1}\leq 1}(X)$.

**Solution.** 1. Let $v \in \mathbb{R}^n$, $\|v\|_1 \leq 1$. We have for $u \in \mathbb{R}^n$:

$$\langle u, v\rangle = \sum_{i=1}^n u_i v_i \leq \sum_{i=1}^n |u_i| \cdot |v_i| \leq \max_{1\leq j\leq n} |u_j| \sum_{i=1}^n |v_i| = \|u\|_\infty \cdot \|v\|_1.$$

This implies that $\langle u, v\rangle - \|u\|_\infty \leq 0$. Since $\langle 0, v\rangle - \|0\|_\infty = 0$ we get

$$\sup_{u\in\mathbb{R}^n} \langle u, v\rangle - \|0\|_\infty = 0.$$

Now let $\|v\|_1 > 1$. Define $u \in \mathbb{R}^n$ with $u_i := \operatorname{sgn}(v_i)$, $1 \leq i \leq n$, which implies $\|u\|_\infty = 1$. We get

$$\langle u, v\rangle = \sum_{i=1}^n |u_i| = \|u\|_1.$$

For $\alpha > 0$ we get

$$\langle \lambda u, v\rangle - \|\alpha u\|_\infty = \alpha \underbrace{(\|u\|_1 - 1)}_{>1}.$$

Therefore

$$\sup_{u\in\mathbb{R}^n} \langle u, v\rangle - \|0\|_\infty = \infty.$$

Altogether we obtain

$$f_1^*(v) = \iota_{\|\cdot\|_1\leq 1}(v).$$

2. We have $f_2 = f_1^*$ and since $f_1$ is closed, proper and convex we have

$$f_2^* = f_1^{**} = f_1.$$

**Exercise 2** (8 Points)**.** Let $A \in \mathbb{R}^{m\times n}$ be a linear operator and $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ a convex function. Then $Af : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defined as

$$(Af)(u) := \begin{cases} \inf_{v\in\mathbb{R}^n,\, Av=u} f(v) & \text{if } \exists\, v \in \mathbb{R}^n \text{ s.t. } Av = u \\ \infty & \text{otherwise.} \end{cases}$$

is called the image of $f$ under $A$.

1. Show that the convex conjugate $(Af)^*$ of $Af$ is given as $f^* \circ A^\top$ where $(f^* \circ A^\top)(v) := f^*(A^\top v)$.
2. Name the properties that we require for $A^\top f^* = (f \circ A)^*$ to hold. What theorem from the lecture applies here?
3. Give an example of a closed, convex and non-empty set $C$ and a linear operator $A$ s.t. $AC := \{Ax : x \in C\}$ is not closed.
4. Let $f$ be closed, (convex) and proper. Argue that $Af$ does not need to be closed.

**Solution.**

1. We find

$$\begin{aligned}(Af)^*(u) &= \sup_{v\in\mathbb{R}^n} \langle u, v\rangle - \inf_{w\in\mathbb{R}^n,\, Aw=v} f(w) \\ &= \sup_{\substack{v\in\mathbb{R}^n \\ w\in\mathbb{R}^n,\, Aw=v}} \langle u, v\rangle - f(w) \\ &= \sup_{w\in\mathbb{R}^n} \langle u, Aw\rangle - f(w) \\ &= \sup_{w\in\mathbb{R}^n} \langle A^\top u, w\rangle - f(w) \\ &= f^*(A^\top u)\end{aligned}$$

2. If $A^\top f^*$ is closed, proper and convex it is equal to its biconjugate and using the result from the previous part we find:

$$A^\top f^* = (A^\top f^*)^{**} = (f \circ A)^*.$$

3. Choose $C := \{x \in \mathbb{R}^2 : x_1 \cdot x_2 \geq 0\}$ and $A := (1, 0)$. Then $C$ is obviously closed but $AC = (0, \infty)$ is open.
4. For injective $A$ we find that the epigraph $\text{epi}(Af)$ of $Af$ is given as

$$\begin{aligned}\text{epi}(Af) &= \{(u, \alpha) \in \mathbb{R}^{n+1} : \exists\, v \in \mathbb{R}^m : Av = u,\ (v, \alpha) \in \text{epi}(f)\} \\ &= \{(Av, \alpha) : (v, \alpha) \in \text{epi}(f)\} = K\text{epi}(f),\end{aligned}$$

for

$$K := \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

**Exercise 3** (4 Points)**.** Let $H : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $R : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ be proper, closed, convex functions and $K \in \mathbb{R}^{m \times n}$ a linear operator. Let there exist a $u \in \mathrm{ri}(\mathrm{dom}(H))$ such that $Ku \in \mathrm{ri}(\mathrm{dom}(R))$. Let $f(u) := H(u) + R(Ku) = \tilde{f}(Au)$, where

$$A := \begin{pmatrix} I \\ K \end{pmatrix} \in \mathbb{R}^{n+m \times n}, \quad \tilde{f}(u, d) := H(u) + R(d).$$

Prove Fenchel's Duality Theorem, i.e. show that

$$\inf_{u \in \mathbb{R}^n} H(u) + R(Ku) = \sup_{q \in \mathbb{R}^m} -H^*(-K^\top q) - R^*(q)$$

Hint: You can assume that the conditions above guarantee that $A^\top \tilde{f}^*$ is closed proper and convex. Argue that $\tilde{f}^*(u, d) = H^*(u) + R^*(d)$. Which result from the lecture applies here? Begin your computation with

$$\inf_{u \in \mathbb{R}^n} f(u) = -\sup_{u \in \mathbb{R}^n} \langle u, 0 \rangle - f(u) = -f^*(0) \ldots$$

**Solution.** Using the result from the lecture for the convex conjugate of a decoupled sum we obtain for $\tilde{f}^*$:

$$\tilde{f}^*(u, d) = H^*(u) + R^*(d).$$

Further we have

$$\begin{aligned}
\inf_{u \in \mathbb{R}^n} H(u) + R(Ku) &= \inf_{u \in \mathbb{R}^n} f(u) \\
&= -\sup_{u \in \mathbb{R}^n} \langle u, 0 \rangle - f(u) \\
&= -f^*(0) \\
&= -(\tilde{f} \circ A)^*(0) \\
&= -(A^\top \tilde{f}^*)(0) \\
&= -\inf_{(q,p),\, q + K^\top p = 0} H^*(q) + R^*(p) \\
&= -\inf_{p \in \mathbb{R}^m} H^*(-K^\top p) + R^*(p) \\
&= \sup_{p \in \mathbb{R}^m} -H^*(-K^\top p) - R^*(p)
\end{aligned}$$

# Programming: Denoising with Duality (Due on 04.12.2017) (12 Points)

**Exercise 4** (12 Points)**.** Denoise the noisy input image $f$, given in the file `noisy_input.png` by solving the dual problem of:

$$\min_u \frac{1}{2} \|u - f\|^2 + \alpha \|Du\|_{2,1}$$

with projected gradient descent. For details of the derivation of the dual problem cf. the lecture.