Convex Optimization for Machine Learning and Computer Vision

Lecture: Dr. Virginia Estellers
Exercises: Emanuel Laude
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 5

Room: 02.09.023

Friday, 02.12.2017, 09:15-11:00

Submission deadline theory: Monday, 27.11.2017, 10:15, Room 02.09.023

Submission deadline coding: Monday, 04.12.2017, 10:15, Room 02.09.023

Theory: Fenchel Duality (10+6 Points)

Exercise 1 (4 Points)**.** Compute the convex conjugates of the following functions:

1. $f_1 : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ where $f_1(X) = \|X\|_{2,\infty}$.
2. $f_2 : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ where $f_2(X) = \delta_{\|\cdot\|_{2,1}\leq 1}(X)$.

Exercise 2 (8 Points)**.** Let $A \in \mathbb{R}^{m\times n}$ be a linear operator and $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ a convex function. Then $Af : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defined as

$$(Af)(u) := \begin{cases} \inf\limits_{v\in\mathbb{R}^n,\, Av=u} f(v) & \text{if } \exists\, v \in \mathbb{R}^n \text{ s.t. } Av = u \\ \infty & \text{otherwise.} \end{cases}$$

is called the image of f under A.

1. Show that the convex conjugate $(Af)^*$ of Af is given as $f^* \circ A^\top$ where $(f^* \circ A^\top)(v) := f^*(A^\top v)$.
2. Name the properties that we require for $A^\top f^* = (f \circ A)^*$ to hold. What theorem from the lecture applies here?
3. Give an example of a closed, convex and non-empty set C and a linear operator A s.t. $AC := \{Ax : x \in C\}$ is not closed.
4. Let f be closed, (convex) and proper. Argue that Af does not need to be closed.

Exercise 3 (4 Points)**.** Let $H : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $R : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ be proper, closed, convex functions and $K \in \mathbb{R}^{m \times n}$ a linear operator. Let there exist a $u \in \text{ri}(\text{dom}(H))$ such that $Ku \in \text{ri}(\text{dom}(R))$. Let $f(u) := H(u) + R(Ku) = \tilde{f}(Au)$, where

$$A := \begin{pmatrix} I \\ K \end{pmatrix} \in \mathbb{R}^{n+m \times n}, \quad \tilde{f}(u, d) := H(u) + R(d).$$

Prove Fenchel's Duality Theorem, i.e. show that

$$\inf_{u \in \mathbb{R}^n} H(u) + R(Ku) = \sup_{q \in \mathbb{R}^m} -H^*(-K^\top q) - R^*(q)$$

Hint: You can assume that the conditions above guarantee that $A^\top \tilde{f}^*$ is closed proper and convex. Argue that $\tilde{f}^*(u, d) = H^*(u) + R^*(d)$. Which result from the lecture applies here? Begin your computation with

$$\inf_{u \in \mathbb{R}^n} f(u) = - \sup_{u \in \mathbb{R}^n} \langle u, 0 \rangle - f(u) = -f^*(0) \ldots$$

Programming: Denoising with Duality (Due on 04.12.2017) (12 Points)

Exercise 4 (12 Points)**.** Denoise the noisy input image f, given in the file `noisy_input.png` by solving the dual problem of:

$$\min_u \frac{1}{2} \|u - f\|^2 + \alpha \|Du\|_{2,1}$$

with projected gradient descent. For details of the derivation of the dual problem cf. the lecture.