Chapter 1
Convex Analysis

Convex Optimization for Machine Learning & Computer Vision
WS 2018/19

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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Convex Set

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Convex Optimization

Notations

- $\mathbb{E}$ is a *Euclidean space* (i.e., finite dimensional inner product space), equipped with
  1. Inner product $\langle \cdot, \cdot \rangle$, e.g., $\langle u, v \rangle = u^\top v$ if $\mathbb{E} = \mathbb{R}^n$;
  2. Norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ satisfying polarization identity:

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2.$$

- C is a closed, convex subset of $\mathbb{E}$.
- J is a convex *objective* function.

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- C is a closed, convex subset of $\mathbb{E}$.
- J is a convex *objective* function.

Convex optimization

$$\text{minimize } J(u) \quad \text{over } u \in C.$$

First questions:

- What is a convex set?
- What is a convex function?

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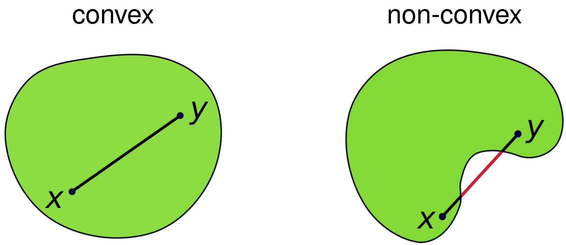
Convex set

Definition

A set C is said to be **convex** if

$$\alpha u + (1-\alpha)v \in C, \quad \forall u, v \in C, \; \forall \alpha \in [0,1].$$



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Recall basic concepts in analysis

Definition

- A set $C \subset \mathbb{E}$ is **open** if $\forall u \in C$, $\exists \epsilon > 0$ s.t. $B_\epsilon(u) \subset C$, where $B_\epsilon(u) := \{v \in \mathbb{E} : \|v - u\| < \epsilon\}$.
- A set $C \subset \mathbb{E}$ is **closed** if its complement $\mathbb{E} \backslash C$ is open.
- The **closure** of a set $C \subset \mathbb{E}$ is

$$\operatorname{cl} C = \{u \in \mathbb{E} : \exists \{u^k\} \subset C \text{ s.t. } \lim_{k \to \infty} u^k = u\}.$$

- The **interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{int} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

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- The **interior** of a set $C \subset \mathbb{E}$ is

$$\mathrm{int}\, C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

- The **relative interior** of a set $C \subset \mathbb{E}$ is

$$\begin{aligned} \mathrm{rint}\, C &:= \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \cap \mathrm{aff}\, C \subset C\} \\ &= \{u \in C : \forall v \in C, \exists \alpha > 1 \text{ s.t. } v + \alpha(u - v) \in C\} \end{aligned}$$

  if C is convex. Here aff C stands for the **affine hull** of C.

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Basic properties

The following operations preserve the convexity:

- Intersection: $C_1 \cap C_2$.
- Summation: $C_1 + C_2 := \{u^1 + u^2 : u^1 \in C_1, u^2 \in C_2\}$.
- Closure: $\operatorname{cl} C$.
- (Relative) interior: $\operatorname{int} C$, $\operatorname{rint} C$.

– In general, the union of convex sets is not convex.

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- (Relative) interior: $\operatorname{int} C$, $\operatorname{rint} C$.

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Convex cone

C is a **cone** if $C = \alpha C$ for any $\alpha > 0$.
C is a **convex cone** if C is a cone and is convex as well.

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Separation of convex sets

Theorem (separation of convex sets)

Let C_1, C_2 be nonempty convex subsets in $\mathbb{E}$.

1. Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E} \backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$
2. Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E} \backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$
3. Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$
4. Assume $\emptyset \neq \operatorname{int} C_1 \subset \mathbb{E} \backslash C_2$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

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Convex functions

- An **extended real-valued function** J maps from $\mathbb{E}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.
- The **domain** of $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{dom} J = \{u \in \mathbb{E} : J(u) < \infty\}.$$

- The function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is **proper** if $\operatorname{dom} J \neq \emptyset$.

Definition

We say $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a **convex function** if

1. $\operatorname{dom} J$ is a convex set.
2. For all $u, v \in \operatorname{dom} J$ and $\alpha \in [0, 1]$ it holds that

$$J(\alpha u + (1 - \alpha)v) \leq \alpha J(u) + (1 - \alpha)J(v).$$

We say J is **strictly convex** if the above inequality is strict for all $\alpha \in (0, 1)$ and $u \neq v$.

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Examples

- $J_{data}(u) = \|u - z\|_q^q$, where $q \geq 1$ and $\|\cdot\|_q$ is ℓ^q-norm.
- $J_{regu}(u) = \|Ku\|_{q'}^{q'}$, where K is linear transform and $q' \geq 1$.
- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$.

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- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$.
- Negative entropy: $J_\epsilon(u) = \epsilon\big(u \log(u) + (1-u)\log(1-u)\big)$.
- Soft plus: $J_\epsilon^*(v) = \epsilon \log(1 + \exp(v/\epsilon)) \xrightarrow{\epsilon \to 0^+} \max(v, 0)$.

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- Negative entropy: $J_\epsilon(u) = \epsilon\big(u \log(u) + (1-u)\log(1-u)\big)$.
- Soft plus: $J_\epsilon^*(v) = \epsilon \log(1 + \exp(v/\epsilon)) \xrightarrow{\epsilon \to 0^+} \max(v, 0)$.
- **Indicator function** ($C \subset \mathbb{E}$ is closed and convex):

$$\delta_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{otherwise.} \end{cases}$$

- Formulate *constrained optimization* with indicator function:

$$\min J(u) \text{ over } u \in C. \quad \Leftrightarrow \quad \min J(u) + \delta_C(u) \text{ over } u \in \mathbb{E}.$$

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Basic facts

(As exercises)

- Any norm (over a normed vector space) is a convex function.
- J is a convex function and A is an affine transform $\Rightarrow J(A\,\cdot)$ is convex function.
- (Jensen's inequality) $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex iff

$$J\left(\sum_{i=1}^{n} \alpha_i u^i\right) \leq \sum_{i=1}^{n} \alpha_i J(u^i),$$

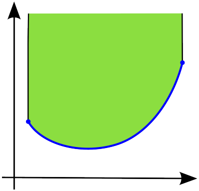
whenever $\{u^i\}_{i=1}^{n} \subset \mathbb{E}$, $\{\alpha_i\}_{i=1}^{n} \subset [0,1]$, $\sum_{i=1}^{n} \alpha_i = 1$.

Epigraph

Definition

The **epigraph** of a proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{epi} J = \{(u, \alpha) \in \mathbb{E} \times \mathbb{R} : J(u) \leq \alpha\}.$$



epigraph

Theorem

A proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex (resp. strictly convex) iff $\operatorname{epi} J$ is a convex (resp. strictly convex) set.

Proof: as exercise.

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Lipschitz continuity

Definition

Assume $J : \mathbb{E} \to \overline{\mathbb{R}}$ with $\operatorname{rint}\operatorname{dom} J \neq \emptyset$. We say J is **locally Lipschitz** at $u \in \operatorname{rint}\operatorname{dom} J$ with modulus $L_u > 0$ if there exists $\epsilon > 0$ s.t.

$$|J(u^1) - J(u^2)| \leq L_u \|u^1 - u^2\| \quad \forall u^1, u^2 \in B_\epsilon(u) \cap \operatorname{rint}\operatorname{dom} J.$$

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Theorem

A proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \operatorname{rint} \operatorname{dom} J$.

Proof: found in script.

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Global vs. Local minimizer

Recall the optimization of $J : \mathbb{E} \to \overline{\mathbb{R}}$:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Definition

1. $u^* \in \mathbb{E}$ is a **global minimizer** if $J(u^*) \leq J(u)$ for all $u \in \mathbb{E}$.
2. u^* is a **local minimizer** if $\exists \epsilon > 0$ s.t. $J(u^*) \leq J(u)$ for all $u \in B_\epsilon(u^*)$.
3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by $J(u^*) < J(u)$.

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3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by $J(u^*) < J(u)$.

Theorem

For any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, if $u^* \in \operatorname{dom} J$ is a local minimizer of J, then it is also a global minimizer.

Proof: on board.

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Does a minimizer always exist?

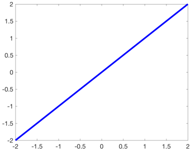
- Consider

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E},$$

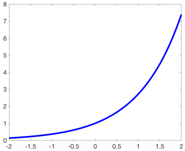
where $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a proper, convex function.

- Some counterexamples for $J : \mathbb{R} \to \overline{\mathbb{R}}$:

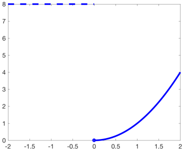
u



$\exp u$



$u^2 + \delta\{u > 0\}$



- Next we formalize our observations and derive sufficient conditions for existence.

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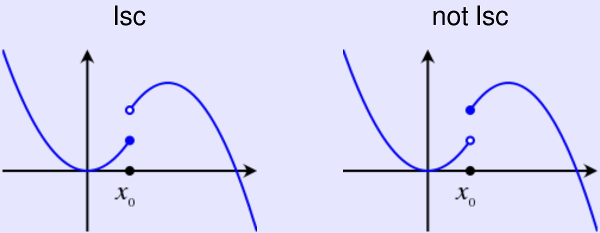
Sufficient conditions for existence

Definition

1. J is **bounded from below** if $J(\cdot) \geq C$ for some $C \in \mathbb{R}$.
2. J is **coercive** if $J(u) \to \infty$ whenever $\|u\| \to \infty$.
   - Proposition: J is coercive if dom J is bounded.
3. J is **lower semi-continuous** (lsc) at u^* if

$$J(u^*) \leq \liminf_{k\to\infty} J(u^k), \text{ whenever } u^k \to u^*.$$



- Proposition: J is lsc iff epi J is closed.

Sufficient conditions for existence

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$$J(u^*) \leq \liminf_{k \to \infty} J(u^k), \text{ whenever } u^k \to u^*.$$

Theorem

Any proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc (everywhere), has a (global) minimizer.

Proof: on board.

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Uniqueness

- Recall that a function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is strictly convex if

$$J(\alpha u + (1-\alpha)v) < \alpha J(u) + (1-\alpha)J(v),$$

for all $u, v \in \operatorname{dom} J,\ u \neq v,\ \alpha \in (0,1)$.

Theorem

The minimizer of a strictly convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is unique.

Proof: on board.

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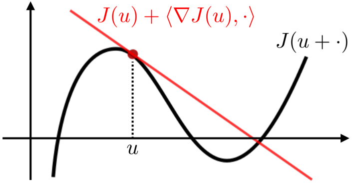
Differential



Definition

$J : \mathbb{E} \to \overline{\mathbb{R}}$ is called (Fréchet) **differentiable** at $u \in \operatorname{int}\operatorname{dom} J$ and $\nabla J(u) \in \mathbb{E}$ is the (Fréchet) **differential** of J at u if

$$\lim_{h \to 0} \frac{|J(u+h) - J(u) - \langle \nabla J(u), h \rangle|}{\|h\|} = 0.$$

J is **continuously differentiable** at $u \in \operatorname{int}\operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $(\operatorname{dom} J) \cap B_\epsilon(u)$ for some $\epsilon > 0$.

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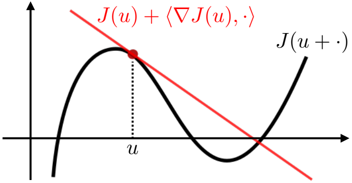
Differential



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Remark

If $\mathbb{E}$ is a topological vector space, $\nabla J(u)$ is treated as a *dual* object in $\mathbb{E}^*$, and $\langle \nabla J(u), h \rangle_{\mathbb{E}^*, \mathbb{E}}$ as *duality pairing*.

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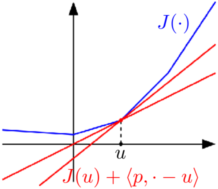
Subdifferential

Definition

The **subdifferential** of a convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ at $u \in \operatorname{dom} J$ is defined by

$$\partial J(u) = \{p \in \mathbb{E} : J(v) \geq J(u) + \langle p, v - u\rangle \ \forall v \in \mathbb{E}\}\,.$$



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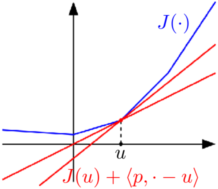

Geometric interpretation

$p \in \partial J(u)$ iff $(p, -1)$ is a normal vector for the supporting hyperplane of $\operatorname{epi} J$ at $(u, J(u))$, i.e.,

$$\left\langle \begin{bmatrix} p \\ -1 \end{bmatrix}, \begin{bmatrix} u \\ J(u) \end{bmatrix} \right\rangle \geq \left\langle \begin{bmatrix} p \\ -1 \end{bmatrix}, \begin{bmatrix} v \\ \alpha \end{bmatrix} \right\rangle, \quad \forall (v, \alpha) \in \operatorname{epi} J.$$

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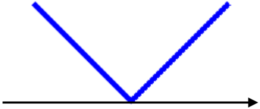
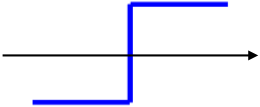
Subdifferential: Examples

1. $J(u) = |u|$.



$J(u)$



$\partial J(u)$

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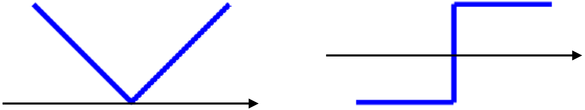

$J(u)$ $\qquad\qquad$ $\partial J(u)$

2. Given a closed, convex subset $C \subset \mathbb{E}$ and $u \in C$,

$$\partial \delta_C(u) = \{p \in \mathbb{E} : \langle p, v - u \rangle \leq 0 \ \forall v \in C\} =: N_C(u),$$

known as the *normal cone* of C at u.

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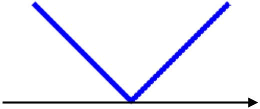
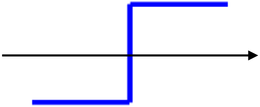
Subdifferential: Examples

1. $J(u) = |u|$.



$J(u)$



$\partial J(u)$

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known as the *normal cone* of C at u.

3. $J(u) = |||u||| \ \Rightarrow \ \partial J(0) = \{p : |||p|||_* \leq 1\}$. $\quad |||\cdot|||_*$ is the *dual norm* of $|||\cdot|||$, i.e., $|||p|||_* = \sup\{\langle p, u \rangle : |||u||| \leq 1\}$.

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Subdifferential calculus

Theorem (chain rule under linear transform)

Let $\widetilde{J}(\cdot) = J(K\cdot)$ with some convex function J and linear transform K. Then

$$\partial \widetilde{J}(u) = K^\top \partial J(Ku)$$

whenever $Ku \in \operatorname{rint} \operatorname{dom} J$.

Example: $J(u) = \|Ku\| \Rightarrow \partial J(u) = K^\top \partial \|\cdot\|(Ku)$.

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Theorem (summation rule)

Let $\widetilde{J}(\cdot) = J_1(\cdot) + J_2(\cdot)$, where J_1, J_2 are convex functions s.t.

$$\operatorname{rint}\operatorname{dom} J_1 \cap \operatorname{rint}\operatorname{dom} J_2 \neq \emptyset.$$

Then for any $u \in \operatorname{dom} J_1 \cap \operatorname{dom} J_2$, we have

$$\partial\widetilde{J}(u) = \partial J_1(u) + \partial J_2(u).$$

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Properties of subdifferential map

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \operatorname{int} \operatorname{dom} J$, $\partial J(u)$ is a nonempty, compact, and convex subset.

Proof: on board.

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Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then ∂J is a **monotone operator**, i.e. $\forall u^1, u^2 \in \operatorname{dom} J,\ p^1 \in \partial J(u^1),\ p^2 \in \partial J(u^2)$:

$$\langle p^1 - p^2, u^1 - u^2 \rangle \geq 0.$$

Proof: on board.

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Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then the set-valued map ∂J is **closed**, i.e. $p^* \in \partial J(u^*)$ whenever

$$\exists (u^k, p^k) \to (u^*, p^*) \in (\operatorname{dom} J) \times \mathbb{E} \ \text{ s.t. } p^k \in \partial J(u^k) \ \ \forall k.$$

Proof: on board.

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Optimality condition

Theorem

Given any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is

$$0 \in \partial J(u^*).$$

Proof: on board.

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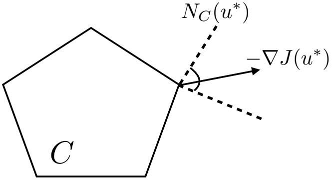
Constrained optimization as special case

If u^* minimizes $\widetilde{J} = J + \delta_C$ with convex function $J : \mathbb{E} \to \mathbb{R}$ and closed convex subset $C \subset \mathbb{E}$, then $0 \in \partial\widetilde{J}(u^*) \Leftrightarrow$

$$0 \in \partial J(u^*) + N_C(u^*).$$



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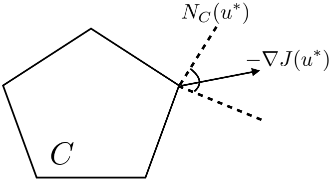

Remark

The optimality condition $0 \in \partial J(u^*) + N_C(u^*)$ is *geometric*. More explicit characterization replies on the *algebraic* representation of $N_C(u^*)$ (e.g., the Karush-Kuhn-Tucker (KKT) conditions) typically under certain *constraint qualifications*.

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Example: Linear-inequality constraints

Let $C = \{u \in \mathbb{R}^n : Au \le b\}$ where $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ has linearly independent rows. Then

$$N_C(u) = \{A^\top \lambda : \lambda \ge 0, \lambda_i = 0 \text{ if } (Au - b)_i < 0\}.$$

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Convex conjugate

Definition

Given a function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the **convex conjugate** (a.k.a. Legendre-Fenchel transform) of J is defined by

$$J^*(p) = \sup_{u \in \mathbb{E}} \{\langle u, p \rangle - J(u)\} \quad \forall p \in \mathbb{E}.$$

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Examples (as exercise)

1. $J(u) = |||u||| \;\Rightarrow\; J^*(p) = \delta\{|||p|||_* \leq 1\}$. $\quad |||\cdot|||_*$ is the *dual norm* of $|||\cdot|||$, i.e., $|||p|||_* = \sup\{\langle p, u \rangle : |||u||| \leq 1\}$.
2. $J(u) = \frac{1}{q}\|u\|_q^q \;\Rightarrow\; J^*(p) = \frac{1}{r}\|p\|_r^r$. $(1 < q < \infty, \frac{1}{q} + \frac{1}{r} = 1)$
3. $J(u) = \sum_{i=1}^n u_i \log u_i + \delta\{u \in \Delta^{n-1}\}$
   $\Rightarrow J^*(p) = \log(\sum_{i=1}^n \exp(p_i))$.

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Basic facts (as exercise)

- Scalar multiplication: $\widetilde{J}(\cdot) = \alpha J(\cdot) \;\Rightarrow\; \widetilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$.
- Translation: $\widetilde{J}(\cdot) = J(\cdot - z) \;\Rightarrow\; \widetilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z \rangle$.

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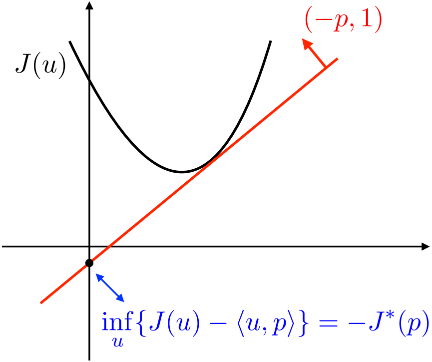
Geometric interpretation

Geometrically, convex conjugation maps

the normal vector of a supporting hyperplane to the epigraph

to

the intersection with the vertical axis.



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Fenchel-Young inequality, order reversing property

Theorem (Fenchel-Young inequality)

For any convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ and $(u, p) \in \mathbb{E} \times \mathbb{E}$, we have

$$J(u) + J^*(p) \geq \langle u, p \rangle .$$

The equality holds iff $p \in \partial J(u)$ with $(u, p) \in \operatorname{dom} J \times \operatorname{dom} J^*$.

Proof: (i) $J(u) + J^*(p) \geq \langle u, p \rangle$ follows directly from the definition of convex conjugate.

(ii) The equality holds only if $(u, p) \in \operatorname{dom} J \times \operatorname{dom} J^*$. Moreover, $p \in \partial J(u)$ is the sufficient and necessary condition for: $\min_{u \in \mathbb{E}} J(u) - \langle u, p \rangle$.

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Theorem (order reversing)

For any $J_1, J_2 : \mathbb{E} \to \overline{\mathbb{R}}$, $J_1^*(\cdot) \leq J_2^*(\cdot)$ whenever $J_1(\cdot) \geq J_2(\cdot)$.

Proof: For any (u, p), we have $\langle u, p \rangle - J_1(u) \leq \langle u, p \rangle - J_2(u)$. Taking supremum over u on both sides yields $J_1^*(p) \leq J_2^*(p)$.

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Biconjugate

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$, and $J^{**} = (J^*)^*$ is the **biconjugate** of J.

In general:

1. $J^{**}(\cdot) \leq J(\cdot)$.
2. J^* is convex and lsc.

If J is proper, convex, and lsc, then:

3. $J^{**}(\cdot) = J(\cdot)$.
4. $p \in \partial J(u)$ iff $u \in \partial J^*(p)$.

Proof: on board.

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Regularity of J and J^*

Definition

1. $J : \mathbb{E} \to \mathbb{R}$ is μ-**strongly convex** if $\exists \mu > 0$ s.t. $J(\cdot) - \frac{\mu}{2}\|\cdot\|^2$ is convex.
2. $J : \mathbb{E} \to \mathbb{R}$ is L-**Lipschitz differentiable** (a.k.a. L-smooth) if J is differentiable and ∇J is Lipschitz with modulus L.

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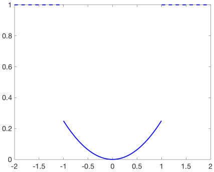
Theorem

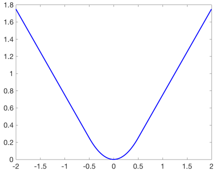
Assume that $J : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, convex, and lsc. Then J is μ-strongly convex iff J^* is $\frac{1}{\mu}$-Lipschitz differentiable.

Proof: on board.

compactly supported quadratic



Huber function



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Fenchel-Rockafellar duality

- Consider

$$\inf_{u \in \mathbb{R}^n} \{F(Ku) + G(u)\},$$

where $K \in \mathbb{R}^{m \times n}$, and F, G are proper, convex, and lsc.

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- The **weak duality** always holds:

$$\begin{aligned}\mathcal{P}^* &:= \inf_u\{F(Ku)+G(u)\}\\ &= \inf_u\sup_p\{\langle p, Ku\rangle - F^*(p)+G(u)\}\\ &\geq \sup_p\inf_u\{\left\langle K^\top p, u\right\rangle + G(u) - F^*(p)\}\\ &= \sup_p\{-G^*(-K^\top p) - F^*(p)\} =: \mathcal{D}^*.\end{aligned}$$

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- Define the **duality gap**:

$$\mathcal{G}(u, p) = F(Ku) + G(u) + G^*(-K^\top p) + F^*(p).$$

Note that $\mathcal{G}(u, p) = 0$ is an optimality criterion.

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Fenchel-Rockafellar duality

- $\mathcal{G}(u^*, p^*) = 0 \Leftrightarrow \mathcal{P}^* = \mathcal{D}^* \Leftrightarrow (u^*, p^*)$ solves the **saddle point problem** with $\mathcal{L}(u, p) := \langle p, Ku \rangle - F^*(p) + G(u)$:

$$\mathcal{L}(u^*, p) \leq \mathcal{L}(u^*, p^*) \leq \mathcal{L}(u, p^*) \quad \forall (u, p).$$

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Theorem (Fenchel-Rockafellar duality)

Assume $\exists \bar{u} \in \operatorname{dom} G$ s.t. F is continuous at $K\bar{u}$. Then the **strong duality** holds: $\mathcal{P}^* = \mathcal{D}^*$. Moreover, (u^*, p^*) is the optimal solution pair iff

$$\begin{cases} Ku^* \in \partial F^*(p^*), \\ -K^\top p^* \in \partial G(u^*). \end{cases}$$

Proof: on board.

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Example: Total-variation image denoising

- Primal problem (given $\Omega \subset \mathbb{R}^d,\ f \in \mathbb{R}^\Omega,\ \alpha > 0,\ q \in [1, \infty]$):

$$\min_{u \in \mathbb{R}^\Omega} \alpha \|\nabla u\|_{1,q} + \frac{1}{2}\|u - f\|^2.$$

Here $\|p\|_{1,q} = \sum_{j \in \Omega} |p_j|_{\ell^q}$ for each $p \in \mathbb{R}^{|\Omega| \times d}$.

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- Apply the Fenchel-Rockafellar duality with

$$F(\cdot) = \alpha\| \cdot \|_{1,q}, \quad K = \nabla, \quad G(\cdot) = \frac{1}{2}\| \cdot - f\|^2.$$

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- Saddle point problem ($1/q + 1/q' = 1$):

$$\max_p \min_u \langle p, \nabla u\rangle - \delta\{\|p\|_{\infty,q'} \le \alpha\} + \frac{1}{2}\|u - f\|^2.$$

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- Dual problem:

$$\min_p \frac{1}{2}\|\nabla^\top p\|^2 + \left\langle \nabla^\top p, f \right\rangle + \delta\{\|p\|_{\infty,q'} \leq \alpha\}.$$

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Proximal operator

Definition

Given a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$, we define the **proximal operator** of J by

$$\operatorname{prox}_{\tau J}(v) = \arg\min_{u} J(u) + \frac{1}{2\tau}\|u - v\|^2.$$

Observations

1. The minimization in prox always has a unique minimizer.
2. By checking the optimality condition,

$$u = \operatorname{prox}_{\tau J}(v) \;\Leftrightarrow\; 0 \in \tau\partial J(u) + u - v \;\Leftrightarrow\; u = (I + \tau\partial J)^{-1}(v).$$

Thus, $\operatorname{prox}_{\tau J} = (I + \tau\partial J)^{-1}$, a.k.a. the **resolvent** of ∂J.

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Proximal Operator

Proximal operator

Definition

Given a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$, we define the **proximal operator** of J by

$$\mathrm{prox}_{\tau J}(v) = \arg\min_u J(u) + \frac{1}{2\tau}\|u - v\|^2.$$

Observations

1. The minimization in prox always has a unique minimizer.
2. By checking the optimality condition,

$$u = \mathrm{prox}_{\tau J}(v) \;\Leftrightarrow\; 0 \in \tau\partial J(u) + u - v \;\Leftrightarrow\; u = (I + \tau\partial J)^{-1}(v).$$

   Thus, $\mathrm{prox}_{\tau J} = (I + \tau\partial J)^{-1}$, a.k.a. the **resolvent** of ∂J.

3. u^* is a **fixed point** of $\mathrm{prox}_{\tau J}$, i.e. $u^* = \mathrm{prox}_{\tau J}(u^*)$,
   $\Leftrightarrow 0 \in \partial J(u^*)$, i.e. u^* is a minimizer of J.

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Proximal operator in optimization algorithms

Proximal gradient

Let us solve the convex optimization:

$$\min_u F(u) + G(u),$$

where G is cont'ly differentiable but F is non-differentiable.
The **proximal gradient** iteration appears as:

$$u^{k+1} = \mathrm{prox}_{\tau F}(u^k - \tau \nabla G(u^k)).$$

Derivation of proximal gradient:

$$\begin{aligned}
& u^* \in \arg\min_u F(u) + G(u) \\
\Leftrightarrow\ & 0 \in \partial F(u^*) + \nabla G(u^*) \\
\Leftrightarrow\ & u^* + \tau \partial F(u^*) \ni u^* - \tau \nabla G(u^*) \\
\Leftrightarrow\ & u^* = (I + \tau \partial F)^{-1}(u^* - \tau \nabla G(u^*)) \\
\rightsquigarrow\ & u^{k+1} = \mathrm{prox}_{\tau F}(u^k - \tau \nabla G(u^k)).
\end{aligned}$$

Examples

1. Indicator function. Let C be nonempty, closed, convex $\Rightarrow$

$$\mathrm{prox}_{\tau\delta_C}(\cdot) = \mathrm{proj}_C(\cdot).$$

In this case: proximal gradient $\Rightarrow$ projected gradient.

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2. Linear approximation. $\widetilde{J}(\cdot) := J(\bar{u}) + \langle \nabla J(\bar{u}), \cdot - \bar{u} \rangle \Rightarrow$

$$\begin{aligned}\operatorname{prox}_{\tau\widetilde{J}}(\bar{u}) &= \arg\min_u \left\{ \frac{1}{2\tau} \|u - \bar{u}\|^2 + \langle \nabla J(\bar{u}), u - \bar{u} \rangle \right\} \\ &= \bar{u} - \tau \nabla J(\bar{u}). \quad \text{(gradient descent step)}\end{aligned}$$

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3. Quadratic approximation.
$$\widetilde{J}(\cdot) := J(\bar{u}) + \langle \nabla J(\bar{u}), \cdot - \bar{u} \rangle + \tfrac{1}{2}\left\langle \nabla^2 J(\bar{u})(\cdot - \bar{u}), \cdot - \bar{u} \right\rangle \Rightarrow$$
$$\begin{aligned}\operatorname{prox}_{\tau\widetilde{J}}(\bar{u}) &= \arg\min_u \left\{ \frac{1}{2\tau}\|u-\bar{u}\|^2 + \langle \nabla J(\bar{u}), u-\bar{u}\rangle \right. \\ &\qquad \left. + \frac{1}{2}\left\langle \nabla^2 J(\bar{u})(u-\bar{u}), u-\bar{u} \right\rangle \right\} \\ &= \bar{u} - \left(\nabla^2 J(\bar{u}) + \frac{1}{\tau} I \right)^{-1} \nabla J(\bar{u}). \quad \text{(damped Newton step)}\end{aligned}$$

Logistic regression (programming exercise)

- MNIST[1] dataset - handwritten digit recognition.
- Train classifier on training set; Evaluate on test set.
- Conv Neural Network: 0.23%; **Logistic Regression**: 10%.

[1] http://yann.lecun.com/exdb/mnist/

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- $C = 10$ classes; grayscale images of pixels $M = 28 \times 28$.
- $N = 60000$ training images $X \in \mathbb{R}^{N \times M}$, with ground-truth labels $Y \in \{1, ..., C\}^N$.

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- Task: Train a *linear classifier* with $W \in \mathbb{R}^{C \times M}$ and $b \in \mathbb{R}^C$,
- ..., which parameterizes *likelihood* via *softmax*:

$$\mathcal{P}(y|x; W, b) = \frac{\exp(\langle W_{y,\cdot}, x\rangle + b_y)}{\sum_{k=1}^{C} \exp(\langle W_{k,\cdot}, x\rangle + b_k)}.$$

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- Minimize: negative log-likelihood $\mathcal{P}$ + regularizations, i.e.,

$$\min_{W,b} \; \mathcal{R}_W(W) + \mathcal{R}_b(b) - \frac{1}{N}\sum_{n=1}^{N} \log \mathcal{P}(Y_n | X_{n,\cdot}; W, b).$$

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Moreau identity

Theorem (Moreau identity)

Let $\tau > 0$ and $J : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. Then the following identity holds:

$$\mathrm{id}(\cdot) = \mathrm{prox}_{\tau J}(\cdot) + \tau\, \mathrm{prox}_{\frac{1}{\tau} J^*}(\cdot/\tau).$$

In particular, $\tau = 1 \Rightarrow \mathrm{id}(\cdot) = \mathrm{prox}_J(\cdot) + \mathrm{prox}_{J^*}(\cdot)$.

<u>Proof</u>:

$$\begin{aligned} & v = \tau\, \mathrm{prox}_{\frac{1}{\tau} J^*}(u/\tau) \\ \Leftrightarrow\ & \left(I + \frac{1}{\tau}\partial J^*\right)^{-1}(u/\tau) = v/\tau \\ \Leftrightarrow\ & \partial J^*(v/\tau) \ni u - v \\ \Leftrightarrow\ & v/\tau \in \partial J(u - v) \\ \Leftrightarrow\ & u - v = (I + \tau\partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u). \end{aligned}$$

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\Leftrightarrow\ & v/\tau \in \partial J(u - v) \\
\Leftrightarrow\ & u - v = (I + \tau\partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u).
\end{aligned}$$

Remark

The Moreau identity suggests that if one of $\mathrm{prox}_J(\cdot)$ and $\mathrm{prox}_{J^*}(\cdot)$ is computable, so is the other.

Infimal convolution

Definition

Let $F, G : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. The **infimal convolution** (or inf convolution) of F and G is defined by

$$(F \Box G)(u) = \inf_{v \in \mathbb{E}} \{F(u - v) + G(v)\},$$

with $\mathrm{dom}(F \Box G) = \mathrm{dom}\, F + \mathrm{dom}\, G$.

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with $\operatorname{dom}(F \Box G) = \operatorname{dom} F + \operatorname{dom} G$.

Theorem

Let $F, G : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. Then

$$(F \Box G)^* = F^* + G^*.$$

Proof: $(F \Box G)^*(p) = \sup_{u,v}\{\langle p, u\rangle - F(v) - G(u - v)\} = \sup_{u,v}\{\langle p, v\rangle - F(v) + \langle p, u - v\rangle - G(u - v)\} = F^*(p) + G^*(p)$.

Analogy to integral convolution

By convolution theorem, $\widehat{F * G} = \widehat{F} \cdot \widehat{G}$ where $\widehat{\ }$ denotes the Fourier transform and $*$ the integral convolution.

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Moreau envelope

Definition

The **Moreau envelope** of a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is defined for each $u \in \mathbb{E}$ by

$$\begin{aligned}\operatorname{env}_{\tau J}(u) &:= \left(J \,\square\, \frac{1}{2\tau}\|\cdot\|^2 \right)(u) \\ &= \inf_{v \in \mathbb{E}} \left\{ J(v) + \frac{1}{2\tau}\|v - u\|^2 \right\} \\ &= J(\operatorname{prox}_{\tau J}(u)) + \frac{1}{2\tau}\|\operatorname{prox}_{\tau J}(u) - u\|^2.\end{aligned}$$

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&= \inf_{v \in \mathbb{E}} \left\{ J(v) + \frac{1}{2\tau} \|v - u\|^2 \right\} \\
&= J(\operatorname{prox}_{\tau J}(u)) + \frac{1}{2\tau} \|\operatorname{prox}_{\tau J}(u) - u\|^2.
\end{aligned}$$

Example

$J : u \mapsto \|u\| \;\Rightarrow\; \operatorname{env}_{\tau J}$ is the *Huber function*:

$$\operatorname{env}_{\tau J}(u) = \begin{cases} \frac{1}{2\tau}\|u\|^2 & \text{if } \|u\| \le \tau, \\ \|u\| - \frac{\tau}{2} & \text{if } \|u\| > \tau. \end{cases}$$

Observation: $\operatorname{env}_{\tau J}$ does smoothing on J.

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Properties of Moreau envelope

- Recall the theorem: $(F \square G)^* = F^* + G^* \Rightarrow$

$$(\mathrm{env}_{\tau J})^* = J^* + \left(\frac{1}{2\tau}\|\cdot\|^2\right)^* = J^* + \frac{\tau}{2}\|\cdot\|^2.$$

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- Recall the theorem: J is μ-strongly convex iff J^* is $\frac{1}{\mu}$-Lipschitz differentiable.

  $\Rightarrow$ $\mathrm{env}_{\tau J}$ is $\frac{1}{\tau}$-Lipschitz differentiable.

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  $\Rightarrow \mathrm{env}_{\tau J}$ is $\frac{1}{\tau}$-Lipschitz differentiable.

- $\nabla \mathrm{env}_{\tau J}$ can be calculated as:

$$\begin{aligned}
p = \nabla \mathrm{env}_{\tau J}(u) \;&\Leftrightarrow\; u \in \partial(\mathrm{env}_{\tau J})^*(p) = \partial J^*(p) + \tau p \\
&\Leftrightarrow\; u - \tau p \in \partial J^*(p) \;\Leftrightarrow\; \partial J(u - \tau p) \ni p \\
&\Leftrightarrow\; \tau \partial J(u - \tau p) \ni \tau p \;\Leftrightarrow\; (I + \tau \partial J)(u - \tau p) \ni u \\
&\Leftrightarrow\; u - \tau p = (I + \tau \partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u) \\
&\Leftrightarrow\; \nabla \mathrm{env}_{\tau J}(u) = p = \frac{1}{\tau}(u - \mathrm{prox}_{\tau J}(u)).
\end{aligned}$$

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