Chapter 2
Optimization Algorithms

Convex Optimization for Machine Learning & Computer Vision
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Gradient-based Methods

Overview of this section

Unconstrained, differentiable, possibly nonconvex optimization

Problem setting:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Assume:

1. $J : \mathbb{E} \to \mathbb{R}$ is continuously differentiable.
2. There exists a global minimizer u^*. (Typically, an optim algorithm seeks for a local minimizer s.t. $\nabla J(u^*) = 0$.)

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Methods under consideration:

1. (Scaled) gradient descent.
2. Line search method.
3. Majorize-minimize method.

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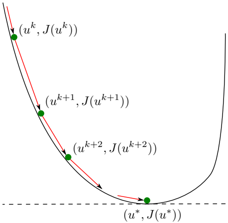
Analytical questions:

1. Convergence (or not); global vs. local convergence.
2. Convergence rate (in special cases).

Descent method



Descent method

Initialize $u^0 \in \mathbb{E}$. Iterate with $k = 0, 1, 2, ...$

1. If the stopping criteria $\|\nabla J(u^k)\| \leq \epsilon$ is *not* satisfied, then continue; otherwise return u^k and stop.
2. Choose a **descent direction** $d^k \in \mathbb{E}$ s.t.

$$\left\langle \nabla J(u^k), d^k \right\rangle < 0.$$

3. Choose an "appropriate" step size $\tau^k > 0$, and update

$$u^{k+1} = u^k + \tau^k d^k.$$

Descent direction

Theorem

If $\langle \nabla J(u^k), d^k \rangle < 0$, then $J(u^k + \tau d^k) < J(u^k)$ for all sufficiently small $\tau > 0$.

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Proof: Use the Taylor expansion:

$$J(u^k + \tau d^k) = J(u^k) + \tau \left\langle \nabla J(u^k), d^k \right\rangle + o(\tau)$$

$$= J(u^k) + \tau \left(\left\langle \nabla J(u^k), d^k \right\rangle + o(1) \right) < J(u^k) \quad \text{as } \tau \to 0^+.$$

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Choices of descent direction

1. Scaled gradient: $d^k = -(H^k)^{-1} \nabla J(u^k)$.
2. Gradient/Steepest descent: $H^k = I$.
3. Newton: $H^k = \nabla^2 J(u^k)$, assuming J is twice continuously differentiable and $\nabla^2 J(u^k)$ is spd.
4. Quasi-Newton: $H^k \approx \nabla^2 J(u^k)$, H^k is spd.

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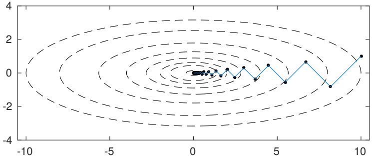
Gradient descent with exact line search



- Gradient descent with *exact* line search:

$$u^{k+1} = u^k - \tau^k \nabla J(u^k),$$
$$\tau^k = \arg\min_{\tau \geq 0} J(u^k - \tau \nabla J(u^k)).$$

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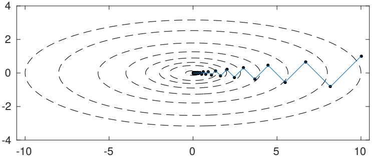

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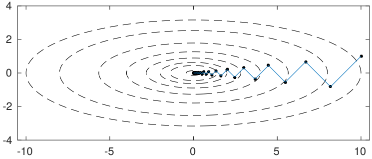
- Case study: $J(u) = \frac{1}{2}\langle u, Qu \rangle - \langle b, u \rangle$, matrix Q is spd.
  - $\nabla J(u) = Qu - b, \quad \| \cdot \|_Q^2 \equiv \langle \cdot, Q \cdot \rangle.$

Gradient descent with exact line search



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  - $\nabla J(u) = Qu - b, \quad \|\cdot\|_Q^2 \equiv \langle \cdot, Q\cdot\rangle$.
  - $\tau^k = \arg\min_{\tau\geq 0} J(u^k - \tau\nabla J(u^k)) = \frac{\|\nabla J(u^k)\|^2}{\|\nabla J(u^k)\|_Q^2} \quad \Rightarrow$

$$\|u^{k+1} - u^*\|_Q^2 = \left(1 - \frac{\|\nabla J(u^k)\|^4}{\|\nabla J(u^k)\|_Q^2 \|\nabla J(u^k)\|_{Q^{-1}}^2}\right)\|u^k - u^*\|_Q^2$$
$$\leq \left(\frac{\lambda_{\max}(Q) - \lambda_{\min}(Q)}{\lambda_{\max}(Q) + \lambda_{\min}(Q)}\right)^2 \|u^k - u^*\|_Q^2.$$

Inexact line search

Backtracking line search

- Sufficient decrease condition (let $c_1 \in (0, 1)$):

$$J(u^k + \tau d^k) \leq J(u^k) + c_1 \tau \left\langle \nabla J(u^k), d^k \right\rangle . \qquad \text{(A)}$$

- Curvature condition (let $c_2 \in (c_1, 1)$):

$$\left\langle \nabla J(u^k + \tau d^k), d^k \right\rangle \geq c_2 \left\langle \nabla J(u^k), d^k \right\rangle . \qquad \text{(C)}$$

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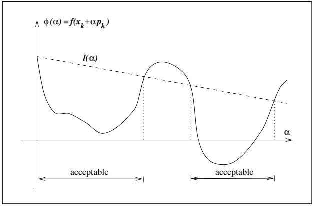
- (A) $\rightsquigarrow$ **Armijo** line search; (A) & (C) $\rightsquigarrow$ **Wolfe-Powell** l.s.

Armijo l.s.



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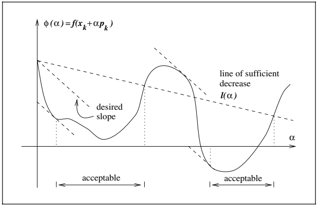

Convergence of backtracking line search

Lemma (feasibility of line search)

Assume that $J : \mathbb{E} \to \mathbb{R}$ is continuously differentiable, $\langle \nabla J(u^k), d^k \rangle < 0 \ \forall k$, and $0 < c_1 < c_2 < 1$. Then there exists an open interval in which the step size τ satisfies (A) and (C).

Proof: on board.

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Proof: on board.

Theorem (Zoutendijk)

Assume that $J : \mathbb{E} \to \mathbb{R}$ is cont'ly differentiable, and (A) and (C) are both satisfied with $0 < c_1 < c_2 < 1$ for each k. In addition, J is μ-Lipschitz differentiable on $\{u \in \mathbb{E} : J(u) \leq J(u^0)\}$. Then

$$\sum_{k=0}^{\infty} \frac{\left| \langle \nabla J(u^k), d^k \rangle \right|^2}{\|d^k\|^2} < \infty.$$

Proof: on board.

Remark

If $\dfrac{\left| \langle \nabla J(u^k), d^k \rangle \right|}{\|\nabla J(u^k)\| \|d^k\|} \geq$ constant > 0, then $\lim_{k \to \infty} \|\nabla J(u^k)\| = 0$.

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Majorize-minimize method

Majorizing function

A function $\widehat{J}(\cdot\,; u)$ is a **majorant** of J at $u \in \mathbb{E}$ if

$$\begin{cases} \widehat{J}(u; u) = J(u), \\ \widehat{J}(\cdot\,; u) \geq J(\cdot). \end{cases}$$

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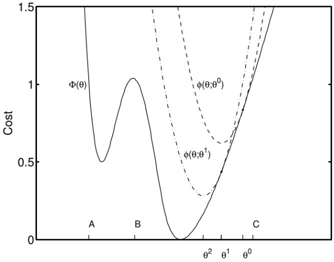
Majorize-minimize (MM) algorithm

Let $\widehat{J}(\cdot; u)$ majorize J $\forall u \in \mathbb{E}$. Then the MM iteration reads:

$$u^{k+1} \in \arg\min_u \widehat{J}(u; u^k).$$



Gradient descent as MM

Remark

1. Monotonic decrease of objectives:

$$J(u^{k+1}) \leq \widehat{J}(u^{k+1}; u^k) \leq \widehat{J}(u^k; u^k) = J(u^k).$$

2. Efficiency of MM relies on the choice of the majorant $\widehat{J}(\cdot; u)$, i.e., $\widehat{J}(\cdot; u)$ is easy to minimize.
3. Common choices of $\widehat{J}(\cdot; u)$ are quadratics.

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Gradient descent as MM

- Observe that $u^{k+1} = u^k - \tau \nabla J(u^k)$ iff

$$u^{k+1} = \arg\min_u J(u^k) + \left\langle \nabla J(u^k), u - u^k \right\rangle + \frac{1}{2\tau} \|u - u^k\|^2.$$

- When $J(u^k) + \langle \nabla J(u^k), \cdot - u^k \rangle + \frac{1}{2\tau} \| \cdot - u^k\|^2 \geq J(\cdot)$ holds?

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Gradient descent as MM

Lemma

Assume that $J : \mathbb{E} \to \mathbb{R}$ is μ-Lipschitz differentiable. Then $\forall u, v \in \mathbb{E}$:

$$|J(v) - J(u) - \langle \nabla J(u), v - u \rangle| \leq \frac{\mu}{2} \|v - u\|^2.$$

Proof: on board.

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Recipe of convergence

By solving the surrogate problem in MM, we achieve: (1) sufficient decrease in the objective; (2) inexact optimality condition which matches the exact OC in the limit.

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Proximal Algorithms

Agenda for the rest of the chapter

- Proximal algorithms for convex optimization:
  - Forward-backward splitting (FBS) / proximal gradient method.
  - Alternating direction method of multipliers (ADMM).
  - Primal-dual hybrid gradient (PDHG).
  - Douglas-Rachford splitting (DRS), Peaceman-Rachford splitting (PRS).

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- Application on examples.
- Connections between algorithms.
- (Unified) convergence analysis.
- Acceleration techniques.

Forward-backward splitting

- Consider

$$\min_u F(u) + G(u),$$

whose minimizer is characterized by

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- **Forward-backward splitting** (FBS):

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau F}(u^k - \tau \nabla G(u^k)) \\ &= (I + \tau\partial F)^{-1} \circ (I - \tau \nabla G)(u^k). \end{aligned}$$

- FBS as *semi-implicit Euler scheme*:

$$\frac{u^{k+1} - u^k}{\tau} \in -\partial F(u^{k+1}) - \nabla G(u^k).$$

Example: Split feasibility problem

Split feasibility problem

Given nonempty, closed, convex sets $C_1 \subset \mathbb{E}_1$, $C_2 \subset \mathbb{E}_2$, and linear operator $K : \mathbb{E}_1 \to \mathbb{E}_2$, find $u \in \mathbb{E}_1$ s.t. $u \in C_1, \; Ku \in C_2$.

- Variational model:

$$\min_{u \in \mathbb{E}_1} \delta_{C_1}(u) + \frac{1}{2}\|Ku - \mathrm{proj}_{C_2}(Ku)\|^2.$$

Note that $\frac{1}{2}\|v - \mathrm{proj}_{C_2}(v)\|^2 = \mathrm{env}_1\, \delta_{C_2}(v)$.

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- Optimality condition:

$$0 \in \partial \delta_{C_1}(u) + K^\top (I - \mathrm{proj}_{C_2})(Ku).$$

  Recall that $\nabla\, \mathrm{env}_1\, \delta_{C_2}(v) = (I - \mathrm{prox}_{\delta_{C_2}})(v)$.

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- Apply FBS $\Rightarrow$

$$\begin{aligned} u^{k+1} &= (I + \tau \partial \delta_{C_1})^{-1}(u^k - \tau K^\top (I - \operatorname{proj}_{C_2})(Ku^k)) \\ &= \operatorname{proj}_{C_1}(u^k - \tau K^\top (I - \operatorname{proj}_{C_2})(Ku^k)). \end{aligned}$$

Example: Regularized least squares

Regularized least squares

$$\min_u F(u) + \frac{1}{2}\|A(u) - b\|^2,$$

where

- A: differentiable operator (modeling the *forward* process).
- b: observation.
- F: regularization/prior term.
  - $\text{prox}_{\tau F}$ is easy to compute.
  - e.g., $F(\cdot) = \|\cdot\|_2^2$, $F(\cdot) = \|\cdot\|_1$, or $F(\cdot) = \|\cdot\|_{\text{nuclear}}$.

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- Optimality condition:

$$0 \in \partial F(u) + \nabla A(u)^\top (A(u) - b).$$

- Apply FBS $\Rightarrow$

$$u^{k+1} = \text{prox}_{\tau F}(u^k - \tau \nabla A(u^k)^\top (A(u^k) - b)).$$