Chapter 2
Optimization Algorithms

Convex Optimization for Machine Learning & Computer Vision
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Gradient-based Methods

Overview of this section

Unconstrained, differentiable, possibly nonconvex optimization

Problem setting:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Assume:

1. $J : \mathbb{E} \to \mathbb{R}$ is continuously differentiable.
2. There exists a global minimizer u^*. (Typically, an optim algorithm seeks for a local minimizer s.t. $\nabla J(u^*) = 0$.)

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Methods under consideration:

1. (Scaled) gradient descent.
2. Line search method.
3. Majorize-minimize method.

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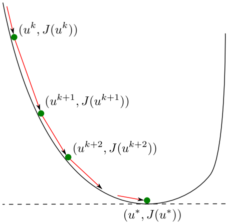
Analytical questions:

1. Convergence (or not); global vs. local convergence.
2. Convergence rate (in special cases).

Descent method



Descent method

Initialize $u^0 \in \mathbb{E}$. Iterate with $k = 0, 1, 2, ...$

1. If the stopping criteria $\|\nabla J(u^k)\| \leq \epsilon$ is *not* satisfied, then continue; otherwise return u^k and stop.
2. Choose a **descent direction** $d^k \in \mathbb{E}$ s.t.
$$\left\langle \nabla J(u^k), d^k \right\rangle < 0.$$
3. Choose an "appropriate" step size $\tau^k > 0$, and update
$$u^{k+1} = u^k + \tau^k d^k.$$

Descent direction

Theorem

If $\langle \nabla J(u^k), d^k \rangle < 0$, then $J(u^k + \tau d^k) < J(u^k)$ for all sufficiently small $\tau > 0$.

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Proof: Use the Taylor expansion:

$$J(u^k + \tau d^k) = J(u^k) + \tau \left\langle \nabla J(u^k), d^k \right\rangle + o(\tau)$$
$$= J(u^k) + \tau \left(\left\langle \nabla J(u^k), d^k \right\rangle + o(1) \right) < J(u^k) \quad \text{as } \tau \to 0^+.$$

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Choices of descent direction

1. Scaled gradient: $d^k = -(H^k)^{-1} \nabla J(u^k)$.
2. Gradient/Steepest descent: $H^k = I$.
3. Newton: $H^k = \nabla^2 J(u^k)$, assuming J is twice continuously differentiable and $\nabla^2 J(u^k)$ is spd.
4. Quasi-Newton: $H^k \approx \nabla^2 J(u^k)$, H^k is spd.

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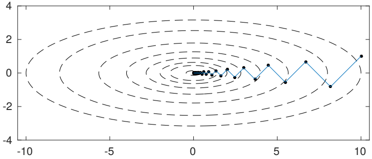
Gradient descent with exact line search



- Gradient descent with *exact* line search:

$$u^{k+1} = u^k - \tau^k \nabla J(u^k),$$
$$\tau^k = \arg\min_{\tau \geq 0} J(u^k - \tau \nabla J(u^k)).$$

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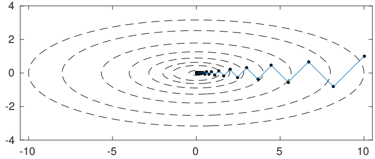

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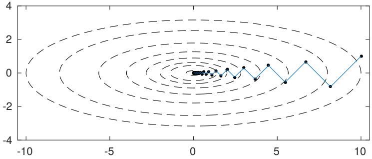
- Case study: $J(u) = \frac{1}{2} \langle u, Qu \rangle - \langle b, u \rangle$, matrix Q is spd.
  - $\nabla J(u) = Qu - b, \;\; \| \cdot \|_Q^2 \equiv \langle \cdot, Q \cdot \rangle.$

Gradient descent with exact line search



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  - $\nabla J(u) = Qu - b, \quad \|\cdot\|_Q^2 \equiv \langle \cdot, Q\cdot\rangle$.
  - $\tau^k = \arg\min_{\tau\geq 0} J(u^k - \tau\nabla J(u^k)) = \dfrac{\|\nabla J(u^k)\|^2}{\|\nabla J(u^k)\|_Q^2} \quad \Rightarrow$

$$\|u^{k+1} - u^*\|_Q^2 = \left(1 - \frac{\|\nabla J(u^k)\|^4}{\|\nabla J(u^k)\|_Q^2 \|\nabla J(u^k)\|_{Q^{-1}}^2}\right)\|u^k - u^*\|_Q^2$$
$$\leq \left(\frac{\lambda_{\max}(Q) - \lambda_{\min}(Q)}{\lambda_{\max}(Q) + \lambda_{\min}(Q)}\right)^2 \|u^k - u^*\|_Q^2.$$

Inexact line search

Backtracking line search

- Sufficient decrease condition (let $c_1 \in (0, 1)$):

$$J(u^k + \tau d^k) \leq J(u^k) + c_1 \tau \left\langle \nabla J(u^k), d^k \right\rangle . \qquad \text{(A)}$$

- Curvature condition (let $c_2 \in (c_1, 1)$):

$$\left\langle \nabla J(u^k + \tau d^k), d^k \right\rangle \geq c_2 \left\langle \nabla J(u^k), d^k \right\rangle . \qquad \text{(C)}$$

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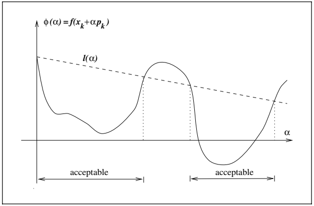
- (A) $\rightsquigarrow$ **Armijo** line search; (A) & (C) $\rightsquigarrow$ **Wolfe-Powell** l.s.

Armijo l.s.



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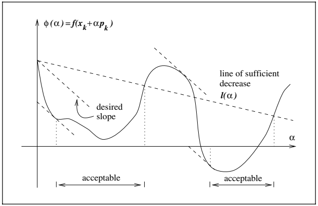

Convergence of backtracking line search

Lemma (feasibility of line search)

Assume that $J : \mathbb{E} \to \mathbb{R}$ is continuously differentiable, $\langle \nabla J(u^k), d^k \rangle < 0 \; \forall k$, and $0 < c_1 < c_2 < 1$. Then there exists an open interval in which the step size τ satisfies (A) and (C).

<u>Proof</u>: on board.

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Proof: on board.

Theorem (Zoutendijk)

Assume that $J : \mathbb{E} \to \mathbb{R}$ is cont'ly differentiable, and (A) and (C) are both satisfied with $0 < c_1 < c_2 < 1$ for each k. In addition, J is μ-Lipschitz differentiable on $\{u \in \mathbb{E} : J(u) \leq J(u^0)\}$. Then

$$\sum_{k=0}^{\infty} \frac{\left|\langle \nabla J(u^k), d^k \rangle\right|^2}{\|d^k\|^2} < \infty.$$

Proof: on board.

Remark

If $\dfrac{\left|\langle \nabla J(u^k), d^k \rangle\right|}{\|\nabla J(u^k)\| \|d^k\|} \geq \text{constant} > 0$, then $\lim_{k\to\infty} \|\nabla J(u^k)\| = 0$.

Majorize-minimize method

Majorizing function

A function $\widehat{J}(\cdot\,;u)$ is a **majorant** of J at $u \in \mathbb{E}$ if

$$\begin{cases} \widehat{J}(u;u) = J(u), \\ \widehat{J}(\cdot\,;u) \geq J(\cdot). \end{cases}$$

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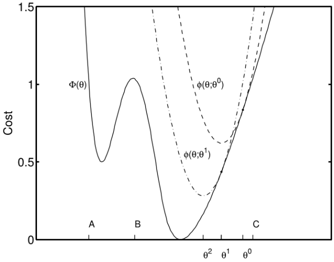
Majorize-minimize (MM) algorithm

Let $\widehat{J}(\cdot; u)$ majorize J $\forall u \in \mathbb{E}$. Then the MM iteration reads:

$$u^{k+1} \in \arg\min_u \widehat{J}(u; u^k).$$



Gradient descent as MM

Remark

1. Monotonic decrease of objectives:

$$J(u^{k+1}) \leq \widehat{J}(u^{k+1}; u^k) \leq \widehat{J}(u^k; u^k) = J(u^k).$$

2. Efficiency of MM relies on the choice of the majorant $\widehat{J}(\cdot; u)$, i.e., $\widehat{J}(\cdot; u)$ is easy to minimize.
3. Common choices of $\widehat{J}(\cdot; u)$ are quadratics.

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Gradient descent as MM

- Observe that $u^{k+1} = u^k - \tau \nabla J(u^k)$ iff

$$u^{k+1} = \arg\min_u J(u^k) + \left\langle \nabla J(u^k), u - u^k \right\rangle + \frac{1}{2\tau}\|u - u^k\|^2.$$

- When $J(u^k) + \langle \nabla J(u^k), \cdot - u^k \rangle + \frac{1}{2\tau}\| \cdot - u^k\|^2 \geq J(\cdot)$ holds?

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Gradient descent as MM

Lemma

Assume that $J : \mathbb{E} \to \mathbb{R}$ is μ-Lipschitz differentiable. Then $\forall u, v \in \mathbb{E}$:

$$|J(v) - J(u) - \langle \nabla J(u), v - u \rangle| \leq \frac{\mu}{2} \|v - u\|^2.$$

Proof: on board.

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Theorem (convergence of gradient descent)

Assume that $J : \mathbb{E} \to \mathbb{R}$ is μ-Lipschitz differentiable. Then the gradient descent iteration

$$u^{k+1} = u^k - \tau \nabla J(u^k)$$

with $\tau \in (0, 1/\mu]$ yields $\lim_{k \to \infty} \nabla J(u^k) = 0$.

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Recipe of convergence

By solving the surrogate problem in MM, we achieve: (1) sufficient decrease in the objective; (2) inexact optimality condition which matches the exact OC in the limit.

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Proximal Algorithms

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Agenda for the rest of the chapter

- Proximal algorithms for convex optimization:
    - Forward-backward splitting (FBS) / proximal gradient method.
    - Alternating direction method of multipliers (ADMM).
    - Primal-dual hybrid gradient (PDHG).
    - Douglas-Rachford splitting (DRS), Peaceman-Rachford splitting (PRS).

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- Application on examples.
- Connections between algorithms.
- (Unified) convergence analysis.
- Acceleration techniques.

Forward-backward splitting

- Consider the convex optimization problem:

$$\min_u F(u) + G(u),$$

with G differentiable and F possibly non-differentiable.

- Its minimizer is characterized by

$$0 \in \partial F(u) + \nabla G(u).$$

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- **Forward-backward splitting** (FBS):

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau F}(u^k - \tau \nabla G(u^k)) \\ &= (I + \tau \partial F)^{-1} \circ (I - \tau \nabla G)(u^k). \end{aligned}$$

- FBS as *semi-implicit Euler scheme*:

$$\frac{u^{k+1} - u^k}{\tau} \in -\partial F(u^{k+1}) - \nabla G(u^k).$$

Example: Split feasibility problem

Split feasibility problem

Given nonempty, closed, convex sets $C_1 \subset \mathbb{E}_1$, $C_2 \subset \mathbb{E}_2$, and linear operator $K : \mathbb{E}_1 \to \mathbb{E}_2$, find $u \in \mathbb{E}_1$ s.t. $u \in C_1,\ Ku \in C_2$.

- Variational model:

$$\min_{u \in \mathbb{E}_1} \delta_{C_1}(u) + \frac{1}{2}\|Ku - \operatorname{proj}_{C_2}(Ku)\|^2.$$

Note that $\frac{1}{2}\|v - \operatorname{proj}_{C_2}(v)\|^2 = \operatorname{env}_1 \delta_{C_2}(v)$.

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- Optimality condition:

$$0 \in \partial \delta_{C_1}(u) + K^\top (I - \mathrm{proj}_{C_2})(Ku).$$

Recall that $\nabla\, \mathrm{env}_1\, \delta_{C_2}(v) = (I - \mathrm{prox}_{\delta_{C_2}})(v)$.

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- Apply FBS $\Rightarrow$

$$\begin{aligned} u^{k+1} &= (I + \tau \partial \delta_{C_1})^{-1}(u^k - \tau K^\top (I - \text{proj}_{C_2})(Ku^k)) \\ &= \text{proj}_{C_1}(u^k - \tau K^\top (I - \text{proj}_{C_2})(Ku^k)). \end{aligned}$$

Demo: Total-variation denoising

Primal model:

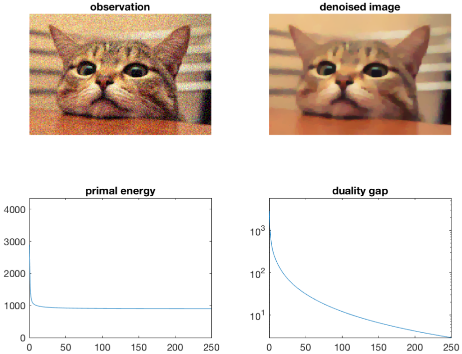
$$\min_u \alpha\|\nabla u\|_1 + \frac{1}{2}\|u-f\|^2.$$

Dual formulation:

$$\min_p \frac{1}{2}\|\nabla^\top p\|^2 + \langle \nabla^\top p, f\rangle + \delta\{\|p\|_\infty \le \alpha\}.$$

Apply FBS on dual (with step size $0 < \tau < 2\|\nabla\|^2$):

$$p^{k+1} = \mathrm{proj}_{\|\cdot\|_\infty \le \alpha}(p^k - \tau\nabla(\nabla^\top p^k + f)).$$



Alternating direction method of multipliers

- Consider

$$\min_{u,v} \; J(u,v) = F(v) + G(u) + \delta\{Ku - v = 0\},$$

given proper, convex, lsc functions F, G and matrix K.

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given proper, convex, lsc functions F, G and matrix K.

- *Augmented Lagrangian* ($\tau > 0$):

$$\mathcal{L}_\tau(u,v;p) = F(v) + G(u) + \langle p, Ku - v\rangle + \frac{\tau}{2}\|Ku - v\|^2,$$

such that

$$\min_{u,v} J(u,v) = \inf_{u,v}\sup_{p}\mathcal{L}_\tau(u,v;p).$$

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- **Alternating direction method of multipliers** (ADMM):

$$\begin{cases} u^{k+1} \in \arg\min_u G(u) + \left\langle p^k, Ku \right\rangle + \frac{\tau}{2}\|Ku - v^k\|^2, \\ v^{k+1} \in \arg\min_v F(v) - \left\langle p^k, v \right\rangle + \frac{\tau}{2}\|Ku^{k+1} - v\|^2, \\ p^{k+1} = p^k + \tau(Ku^{k+1} - v^{k+1}). \end{cases}$$

Example: Consensus ADMM

- **Empirical risk minimization** (ERM):

$$\min_{u} \; F(u) + \frac{1}{n} \sum_{i=1}^{n} G_i(u),$$

where G_i represents the training error on sample (x_i, y_i):

$$G_i(u) = \text{loss}(h(x_i; u), y_i),$$

and F represents the model prior.

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- **Consensus optimization**:

$$\min_{\{u_i\},v} \; F(u) + \frac{1}{n}\sum_{i=1}^{n} G_i(v_i)$$
$$\text{s.t.} \;\; v_i = u \quad \forall i \in \{1, ..., n\}.$$

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- Augmented Lagrangian:

$$\mathcal{L}_\tau(u, \{v_i\}, \{p_i\}) = F(u) + \frac{1}{n}\sum_{i=1}^{n}\left(G_i(v_i) + \langle p_i, v_i - u\rangle + \frac{\tau}{2}\|v_i - u\|^2\right).$$

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- **Consensus ADMM**:

$$u^{k+1} = \operatorname{prox}_{F/\tau}\left(\frac{1}{n}\sum_{i=1}^{n}\left(v_i^k + \frac{1}{\tau}p_i^k\right)\right),$$
$$\forall i: \;\; v_i^{k+1} = \operatorname{prox}_{G_i/\tau}\left(u^{k+1} - \frac{1}{\tau}p_i^k\right),$$
$$\forall i: \;\; p_i^{k+1} = p_i^k + \tau(v_i^{k+1} - u^{k+1}).$$

Primal-dual hybrid gradient

- By Fenchel-Rockafellar duality theorem, we reformulate

$$\min_u \; F(Ku) + G(u)$$

as the saddle-point problem:

$$\sup_p \inf_u \langle p, Ku \rangle + G(u) - F^*(p).$$

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- By Fenchel-Rockafellar duality theorem, we reformulate

$$\min_u \; F(Ku) + G(u)$$

as the saddle-point problem:

$$\sup_p \inf_u \langle p, Ku \rangle + G(u) - F^*(p).$$

- **Primal-dual hybrid gradient** (PDHG) ($st > \|K\|^2$):

$$u^{k+1} = \arg\min_u \left\langle u, K^\top p^k \right\rangle + G(u) + \frac{s}{2}\|u - u^k\|^2,$$
$$p^{k+1} = \arg\min_p -\left\langle K(2u^{k+1} - u^k), p \right\rangle + F^*(p) + \frac{t}{2}\|p - p^k\|^2.$$

- Optimality conditions for the updates:

$$0 \in \partial G(u^{k+1}) + K^\top p^k + s(u^{k+1} - u^k),$$
$$0 \in \partial F^*(p^{k+1}) - K(2u^{k+1} - u^k) + t(p^{k+1} - p^k).$$

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Scaled primal-dual hybrid gradient

- Recall PDGH:

$$0 \in \partial G(u^{k+1}) + K^\top p^k + s(u^{k+1} - u^k),$$
$$0 \in \partial F^*(p^{k+1}) - K(2u^{k+1} - u^k) + t(p^{k+1} - p^k).$$

- Replace s, t by spd matrices $S, T \rightsquigarrow$ Scaled PDHG:

$$0 \in \partial G(u^{k+1}) + K^\top p^k + S(u^{k+1} - u^k),$$
$$0 \in \partial F^*(p^{k+1}) - K(2u^{k+1} - u^k) + T(p^{k+1} - p^k).$$

- Scaled PDHG in compact form:

$$0 \in \begin{bmatrix} S & -K^\top \\ -K & T \end{bmatrix} \left(\begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix} - \begin{bmatrix} u^k \\ p^k \end{bmatrix} \right) + \begin{bmatrix} \partial G & K^\top \\ -K & \partial F^* \end{bmatrix} \begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix}.$$

Scaled primal-dual hybrid gradient

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- Scaled PDHG is a **customized proximal iteration**:

$$\boxed{0 \in M(\xi^{k+1} - \xi^k) + R(\xi^{k+1})} \Leftrightarrow \boxed{\xi^{k+1} = (M + R)^{-1} M \xi^k}$$

- Sufficient conditions for convergence:
  (1) M is spd matrix; (2) R is maximal monotone operator.

Interpret ADMM as customized proximal iteration

- Recall ADMM (with reordered updates):

$$v^{k+1} \in \arg\min_{v} F(v) - \left\langle p^k, v \right\rangle + \frac{\tau}{2}\|Ku^k - v\|^2, \quad (1)$$

$$p^{k+1} = p^k + \tau(Ku^k - v^{k+1}), \quad (2)$$

$$u^{k+1} \in \arg\min_{u} G(u) + \left\langle p^{k+1}, Ku \right\rangle + \frac{\tau}{2}\|Ku - v^{k+1}\|^2. \quad (3)$$

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Interpret ADMM as customized proximal iteration

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- ADMM as customized proximal iteration:

$$(1) \;\Rightarrow\; 0 \in \partial F(v^{k+1}) - p^k + \tau(v^{k+1} - Ku^k), \quad (4)$$

$$(3) \;\Rightarrow\; 0 \in \partial G(u^{k+1}) + K^\top p^{k+1} + \tau K^\top(Ku^{k+1} - v^{k+1}), \quad (5)$$

$$(2),(4) \;\Rightarrow\; p^{k+1} \in \partial F(v^{k+1}) \;\Leftrightarrow\; v^{k+1} \in \partial F^*(p^{k+1}), \quad (6)$$

$$(2),(5) \;\Rightarrow\; 0 \in \partial G(u^{k+1}) + K^\top(2p^{k+1} - p^k) + \tau K^\top K(u^{k+1} - u^k), \quad (7)$$

$$(2),(6) \;\Rightarrow\; 0 \in -Ku^k + \frac{1}{\tau}(p^{k+1} - p^k) + \partial F^*(p^{k+1}), \quad (8)$$

$$(7),(8) \;\Rightarrow\; 0 \in \begin{bmatrix} \tau K^\top K & K^\top \\ K & \frac{1}{\tau} I \end{bmatrix} \begin{bmatrix} u^{k+1} - u^k \\ p^{k+1} - p^k \end{bmatrix} + \begin{bmatrix} \partial G & K^\top \\ -K & \partial F^* \end{bmatrix} \begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix}.$$

Reflection operator

- Given a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$ and $\tau > 0$, we call

$$\text{refl}_{\tau J} = 2\,\text{prox}_{\tau J} - I = 2(I + \tau \partial J)^{-1} - I$$

the **reflection operator** on ∂J.

- In a more general definition for "refl", ∂J is replaced by a *maximal monotone operator*.
  - We don't formally introduce maximal monotone operator.
  - Fact: For any proper, convex, lsc function J, ∂J is indeed a maximal monotone operator.

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  - We don't formally introduce maximal monotone operator.
  - Fact: For any proper, convex, lsc function J, ∂J is indeed a maximal monotone operator.

- Fixed points of $\operatorname{refl}_{\tau J}$:

$$\begin{aligned} & u = \operatorname{refl}_{\tau J}(u) \\ \Leftrightarrow\ & u = 2\operatorname{prox}_{\tau J}(u) - u \\ \Leftrightarrow\ & u = \operatorname{prox}_{\tau J}(u) \\ \Leftrightarrow\ & 0 \in \partial J(u). \end{aligned}$$

Douglas-Rachford- & Peaceman-Rachford splitting

- Consider the *monotone inclusion* problem:

$$0 \in \partial F(u) + \partial G(u).$$

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Douglas-Rachford- & Peaceman-Rachford splitting

- Consider the *monotone inclusion* problem:

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- **Douglas-Rachford splitting** (DRS):

$$\begin{cases} u^{k+1} = \operatorname{prox}_{\tau G}(v^k), \\ v^{k+1} = v^k - u^{k+1} + \operatorname{prox}_{\tau F}(2u^{k+1} - v^k). \end{cases} \tag{DRS}$$

- **Peaceman-Rachford splitting** (PRS):

$$\begin{cases} u^{k+1} = \operatorname{prox}_{\tau G}(v^k), \\ v^{k+1} = v^k - 2u^{k+1} + 2\operatorname{prox}_{\tau F}(2u^{k+1} - v^k). \end{cases} \tag{PRS}$$

- DRS & PRS in compact forms:

$$v^{k+1} = \left(\frac{1}{2}I + \frac{1}{2}\operatorname{refl}_{\tau F} \circ \operatorname{refl}_{\tau G}\right)(v^k), \tag{DRS'}$$

$$v^{k+1} = (\operatorname{refl}_{\tau F} \circ \operatorname{refl}_{\tau G})(v^k). \tag{PRS'}$$

Douglas-Rachford- & Peaceman-Rachford splitting

Fixed points of DRS & PRS:

$$
\begin{aligned}
& v = \mathrm{refl}_{\tau F}(\mathrm{refl}_{\tau G}(v)) = 2\,\mathrm{prox}_{\tau F}(\mathrm{refl}_{\tau G}(v)) - \mathrm{refl}_{\tau G}(v) \\
\Leftrightarrow\; & \mathrm{prox}_{\tau F}(\mathrm{refl}_{\tau G}(v)) = \mathrm{prox}_{\tau G}(v) \\
\Leftrightarrow\; & \mathrm{refl}_{\tau G}(v) \in (I + \tau\partial F)(\mathrm{prox}_{\tau G}(v)) \\
\Leftrightarrow\; & 2\,\mathrm{prox}_{\tau G}(v) - v \in \mathrm{prox}_{\tau G}(v) + \tau\partial F(\mathrm{prox}_{\tau G}(v)) \\
\Leftrightarrow\; & \mathrm{prox}_{\tau G}(v) - v \in \tau\partial F(\mathrm{prox}_{\tau G}(v)) \\
\Leftrightarrow\; & u = \mathrm{prox}_{\tau G}(v), \quad u - v \in \tau\partial F(u) \\
\Leftrightarrow\; & v \in u + \tau\partial G(u), \quad u - v \in \tau\partial F(u) \\
\Leftrightarrow\; & 0 \in \partial F(u) + \partial G(u).
\end{aligned}
$$

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Interpret DRS as customized proximal iteration

- Apply DRS to: $\min_u F(u) + G(u)$. $\Rightarrow$

$$u^{k+1} = \text{prox}_{\tau G}(v^k), \tag{1}$$
$$v^{k+1} = v^k - u^{k+1} + \text{prox}_{\tau F}(2u^{k+1} - v^k). \tag{2}$$

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$$u^{k+1} = \text{prox}_{\tau G}(v^k), \tag{1}$$
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- DRS as customized proximal iteration ($p^k := (u^k - v^k)/\tau$):

$$\begin{aligned}
(1) \;&\Leftrightarrow\; u^{k+1} = \text{prox}_{\tau G}(u^k - \tau p^k) \;\Leftrightarrow\; u^k - \tau p^k \in (I + \tau \partial G)u^{k+1} \\
&\Leftrightarrow\; 0 \in (u^{k+1} - u^k)/\tau + p^k + \partial G(u^{k+1}), \qquad (3) \\
(2) \;&\Leftrightarrow\; 2u^{k+1} - u^k + \tau p^k = \tau p^{k+1} + \text{prox}_{\tau F}(2u^{k+1} - u^k + \tau p^k) \\
&\Rightarrow\; \tau p^{k+1} = (I - \text{prox}_{\tau F})(2u^{k+1} - u^k + \tau p^k) \\
&\Leftrightarrow\; p^{k+1} = \text{prox}_{\frac{1}{\tau}F^*}((2u^{k+1} - u^k)/\tau + p^k) \quad \text{by Moreau's identity} \\
&\Leftrightarrow\; (2u^{k+1} - u^k)/\tau + p^k \in \left(I + \frac{1}{\tau}\partial F^*\right)(p^{k+1}) \\
&\Leftrightarrow\; 0 \in \tau(p^{k+1} - p^k) + \partial F^*(p^{k+1}) - (2u^{k+1} - u^k), \qquad (4) \\
(3),(4) \;&\Rightarrow\; 0 \in \begin{bmatrix} \frac{1}{\tau}I & -I \\ -I & \tau I \end{bmatrix} \begin{bmatrix} u^{k+1} - u^k \\ p^{k+1} - p^k \end{bmatrix} + \begin{bmatrix} \partial G & I \\ -I & \partial F^* \end{bmatrix} \begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix}.
\end{aligned}$$

Demo: Multiclass segmentation

- Variational model:

$$\min_{u:\Omega\to\Delta^{L-1}} \sum_{j\in\Omega} \Big(\delta\{u_j \in \Delta^{L-1}\} + \langle u_j, f_j\rangle\Big) + \alpha \sum_{l=1}^{L} \|\nabla u^l\|_1,$$

where Δ^{L-1} is the probability simplex in $\mathbb{R}^L$.

- Segmentation results:

image

segmentation ($L = 4$)

Convergence Theory

Fixed-point iteration

Fixed-point iteration

Proximal algorithm as *fixed-point iteration*:

$$u^{k+1} = \Phi(u^k).$$

Its convergence depends on the property of Φ.

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Fixed-point iteration

Fixed-point iteration

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$$u^{k+1} = \Phi(u^k).$$

Its convergence depends on the property of Φ.

Definition

Let C be a nonempty, closed, convex subset of $\mathbb{E}$, and $\Phi : C \to \mathbb{E}$. Then Φ is:

1. μ-Lipschitz with modulus $\mu \geq 0$ if

$$\forall u, v \in C : \|\Phi(u) - \Phi(v)\| \leq \mu \|u - v\|.$$

2. **contractive** if Φ is μ-Lipschitz with modulus $\mu \in [0, 1)$.
3. **nonexpansive** if Φ is 1-Lipschitz.

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Remark

1. If Φ is contractive (mod. $\mu \in [0, 1)$), then by **Banach fixed point theorem** the iteration $u^{k+1} = \Phi(u^k)$ converges to the unique fixed point u^* linearly: $\|u^k - u^*\| \leq \mu^k \|u^0 - u^*\|$.

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2. Unfortunately, Banach fixed point theorem does not apply here. Most proximal algorithms consist of nonexpansive operators Φ (including proj, prox, and refl), which are not contractive but “averaged’ operators”.

Averaged operator

Definition

Let C be a nonempty, closed, convex subset of $\mathbb{E}$, and $\Phi : C \to \mathbb{E}$. Then Φ is α-**averaged** with $\alpha \in (0, 1)$ if there exists a nonexpansive operator $\Psi : C \to \mathbb{E}$ such that

$$\Phi = (1 - \alpha)I + \alpha\Psi.$$

In particular, "$\frac{1}{2}$-averaged" is also called **firmly nonexpansive**.

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Proposition

Let C be a nonempty, closed, convex subset of $\mathbb{E}$, $\Phi : C \to \mathbb{E}$, and $\alpha \in (0,1)$. Then the following statements are equivalent:

1. Φ is α-averaged.
2. $(1-\frac{1}{\alpha})I + \frac{1}{\alpha}\Phi$ is nonexpansive.
3. $\forall u, v \in C : \|\Phi(u)-\Phi(v)\|^2 \leq \|u-v\|^2 - \frac{1-\alpha}{\alpha}\|(I-\Phi)(u)-(I-\Phi)(v)\|^2$.
4. $\forall u, v \in C : \|\Phi(u)-\Phi(v)\|^2 + (1-2\alpha)\|u-v\|^2 \leq 2(1-\alpha)\langle u-v, \Phi(u)-\Phi(v)\rangle$.

Proof: on board.

Averaged operator in proximal algorithms

- Recall the customized proximal iteration:

$$u^{k+1} = \Phi^{(\text{cpi})}(u^k), \quad \Phi^{(\text{cpi})} = (M + R)^{-1} M,$$

  for given spd matrix M and monotone operator R.
    - One can verify that $\Phi^{(\text{cpi})}$ is firmly nonexpansive under the scaled norm $\|\cdot\|_M = \sqrt{\langle \cdot, M \cdot \rangle}$.

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- Recall Douglas-Rachford splitting (in compact form):

$$v^{k+1} = \Phi^{(\text{drs})}(v^k), \quad \Phi^{(\text{drs})} = \frac{1}{2} I + \frac{1}{2} \text{refl}_{\tau F} \circ \text{refl}_{\tau G},$$

  for some proper, convex, lsc functions $F, G : \mathbb{E} \to \overline{\mathbb{R}}$.
  - Since $\text{refl}_{\tau F} = 2 \, \text{prox}_{\tau F} - I$ is nonexpansive and so is $\text{refl}_{\tau G}$, $\Phi^{(\text{drs})}$ is firmly nonexpansive.

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for some proper, convex, lsc functions $F, G : \mathbb{E} \to \overline{\mathbb{R}}$.

  - Since $\operatorname{refl}_{\tau F} = 2 \operatorname{prox}_{\tau F} - I$ is nonexpansive and so is $\operatorname{refl}_{\tau G}$, $\Phi^{(\text{drs})}$ is firmly nonexpansive.

- Recall forward-backward splitting:

$$u^{k+1} = \Phi^{(\text{fbs})}(u^k), \quad \Phi^{(\text{fbs})} = \operatorname{prox}_{\tau F} \circ (I - \tau \nabla G),$$

where G is μ-Lipschitz differentiable and $\tau \in (0, 2/\mu)$.

  - As a consequence of the Baillon-Haddad Theorem (next slide), $I - \tau \nabla G$ is an averaged operator. Hence, $\Phi^{(\text{fbs})}$ is a composition of two averaged operators (again averaged).

Averaged operator in gradient descent

Theorem (Baillon-Haddad)

Let $J : \mathbb{E} \to \mathbb{R}$ be a convex, continuously differentiable function. Then ∇J is a nonexpansive operator iff ∇J is firmly nonexpansive.

Proof: on board.

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Averaged operator in gradient descent

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Let $J : \mathbb{E} \to \mathbb{R}$ be a convex, continuously differentiable function. Then ∇J is a nonexpansive operator iff ∇J is firmly nonexpansive.

Proof: on board.

Corollary

Assume $G : \mathbb{E} \to \mathbb{R}$ is convex and μ-Lipschitz differentiable, and $\tau = 2\alpha/\mu$ with $\alpha \in (0, 1)$. Then $I - \tau \nabla G$ is α-averaged.

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Averaged operator in gradient descent

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Proof: on board.

Corollary

Assume $G : \mathbb{E} \to \mathbb{R}$ is convex and μ-Lipschitz differentiable, and $\tau = 2\alpha/\mu$ with $\alpha \in (0, 1)$. Then $I - \tau \nabla G$ is α-averaged.

Proof: Since $\frac{1}{\mu}\nabla G$ is nonexpansive, by the Baillon-Haddad theorem, $\frac{1}{\mu}\nabla G$ is firmly nonexpansive, i.e., $\exists \Psi : \mathbb{E} \to \mathbb{E}$ nonexpansive s.t. $\frac{1}{\mu}\nabla G = \frac{1}{2} I + \frac{1}{2}\Psi$. Hence,

$$I - \tau \nabla G = (1 - \frac{\tau\mu}{2}) I - \frac{\tau\mu}{2}\Psi = (1 - \alpha) I + \alpha(-\Psi),$$

i.e. $I - \tau \nabla G$ is α-averaged.

Composition of averaged operators

In forward-backward splitting,

$$\Phi^{(\mathrm{fbs})} = \mathrm{prox}_{\tau F} \circ \left(I - \frac{2\alpha}{\mu} \nabla G \right)$$

appears as the composition of a $\frac{1}{2}$-averaged operator $\mathrm{prox}_{\tau F}$ and an α-averaged operator $I - \frac{2\alpha}{\mu} \nabla G$ with $\alpha \in (0, 1)$.

Composition of averaged operators

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Theorem (composition of averaged operators)

Let C be a nonempty, closed, convex subset of $\mathbb{E}$. For each $i \in \{1, ..., m\}$, let $\alpha_i \in (0, 1)$ and $\Phi_i : C \to C$ be an α_i-averaged operator. Then

$$\Phi = \Phi_m \circ ... \circ \Phi_1$$

is α-averaged with

$$\alpha = \frac{m}{m - 1 + \frac{1}{\max_{1 \le i \le m} \alpha_i}}.$$

Proof: on board.

Convex combination of averaged operators

Theorem (convex combination of averaged operators)

Let C be a nonempty, closed, convex subset of $\mathbb{E}$. For each $i \in \{1, ..., m\}$, let $\alpha_i \in (0, 1)$, $\omega_i \in (0, 1)$ and $\Phi_i : C \to \mathbb{E}$ be an α_i-averaged operator. If $\sum_{i=1}^{m} \omega_i = 1$ and $\alpha = \max_{1 \leq i \leq m} \alpha_i$, then

$$\Phi = \sum_{i=1}^{m} \omega_i \Phi_i$$

is α-averaged.

Proof: as exercise.

Convergence of averaged-operator iterations

Theorem (Krasnoselskii)

Let C be a nonempty, closed, convex subset of $\mathbb{E}$, and $u^{k+1} = \Phi(u^k)$ for $k = 0, 1, 2, \ldots$ where $\Phi : C \to C$ satisfies:

1. Φ is α-averaged for some $\alpha \in (0, 1)$.
2. Φ has at least one fixed point.

Then $\{u^k\}$ converges to a fixed point of Φ.

Proof: on board.

Convergence of averaged-operator iterations

Theorem (Krasnoselskii-Mann)

Let C be a nonempty, closed, convex subset of $\mathbb{E}$, and $u^{k+1} = (1-\tau^k)u^k + \tau^k\Psi(u^k)$ for $k = 0, 1, 2, \ldots$ where $\{\tau^k\} \subset [0, 1]$ s.t.

$$\sum_{k=0}^{\infty} \tau^k(1-\tau^k) = \infty,$$

and $\Psi : C \to C$ satisfies:

1. Ψ is nonexpansive.
2. Ψ has at least one fixed point.

Then $\{u^k\}$ converges to a fixed point of Ψ.

Proof: on board.

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Convergence of averaged-operator iterations

Theorem (Krasnoselskii-Mann)

Let C be a nonempty, closed, convex subset of $\mathbb{E}$, and $u^{k+1} = (1 - \tau^k)u^k + \tau^k \Psi(u^k)$ for $k = 0, 1, 2, \ldots$ where $\{\tau^k\} \subset [0, 1]$ s.t.

$$\sum_{k=0}^{\infty} \tau^k (1 - \tau^k) = \infty,$$

and $\Psi : C \to C$ satisfies:

1. Ψ is nonexpansive.
2. Ψ has at least one fixed point.

Then $\{u^k\}$ converges to a fixed point of Ψ.

Proof: on board.

Remarks

1. Condition $\sum_{k=0}^{\infty} \tau^k (1 - \tau^k) = \infty$ is fulfilled if $\{\tau^k\} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1/2]$.
2. Decay rate of fixed-point residual: $\|u^{k+1} - u^k\| = o(1/\sqrt{k})$.

Convergence in infinite dimensional space

Theorem (Krasnoselskii in Hilbert space)

Let C be a nonempty, closed, convex subset of a (real) Hilbert space $\mathbb{H}$, and $u^{k+1} = \Phi(u^k)$ for $k = 0, 1, 2, ...$ where $\Phi : C \to C$ satisfies:

1. Φ is α-averaged for some $\alpha \in (0, 1)$.
2. Φ has at least one fixed point.

Then $\{u^k\}$ converges *weakly* to a fixed point of Φ.

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Convergence in infinite dimensional space

Theorem (Krasnoselskii in Hilbert space)

Let C be a nonempty, closed, convex subset of a (real) Hilbert space $\mathbb{H}$, and $u^{k+1} = \Phi(u^k)$ for $k = 0, 1, 2, \ldots$ where $\Phi : C \to C$ satisfies:

1. Φ is α-averaged for some $\alpha \in (0, 1)$.
2. Φ has at least one fixed point.

Then $\{u^k\}$ converges *weakly* to a fixed point of Φ.

Proof: ... $\Rightarrow \|u^{k+1} - \bar{u}\|^2 \leq \|u^0 - \bar{u}\|^2 - \frac{1-\alpha}{\alpha} \sum_{l=0}^{k} \|(I - \Phi)(u^l)\|^2$
$\Rightarrow$ (i) $\|u^k - \bar{u}\| \searrow c \geq 0$; (ii) $\sum_{k=0}^{\infty} \|(I - \Phi)(u^k)\|^2 < \infty$.
(i) $\Rightarrow \{u^k\}$ converges weakly to $u^* \in C$ along a subsequence;
(ii) & "demiclosedness principle" $\Rightarrow u^* - \Phi(u^*) = 0$. $\Rightarrow$ ... $\square$

Lemma (demiclosedness principle)

Let C be a nonempty, closed, convex subset of a (real) Hilbert space $\mathbb{H}$, and $\Phi : C \to \mathbb{H}$ be nonexpansive. For any sequence $\{u^k\} \subset C$ s.t. $\{u^k\}$ weakly converges to $u \in C$ and $u^k - \Phi(u^k)$ strongly converges to $v \in \mathbb{H}$, we have $u - \Phi(u) = v$.

Linear convergence under strong monotonicity

- Recall the customized proximal iteration:

$$0 \in M(u^{k+1} - u^k) + R(u^{k+1}),$$

where M is spd matrix, R is (maximal) monotone operator.

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- Let $u^* = \lim_{k\to\infty} u^k$, $0 \in R(u^*)$, and $\xi^{k+1} \in R(u^{k+1})$ s.t.

$$\begin{aligned} 0 &= \left\langle u^{k+1} - u^*, u^{k+1} - u^k \right\rangle_M + \left\langle u^{k+1} - u^*, \xi^{k+1} - 0 \right\rangle \\ &= \frac{1}{2}\|u^{k+1} - u^*\|_M^2 - \frac{1}{2}\|u^k - u^*\|_M^2 + \frac{1}{2}\|u^{k+1} - u^k\|_M^2 \\ &\quad + \left\langle u^{k+1} - u^*, \xi^{k+1} - 0 \right\rangle . \end{aligned}$$

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- Previously, we only assume R is monotone

$$\begin{aligned} &\Rightarrow \ \left\langle u^{k+1} - u^*, \xi^{k+1} - 0 \right\rangle \geq 0 \\ &\Rightarrow \ \frac{1}{2}\|u^{k+1} - u^*\|_M^2 \leq \frac{1}{2}\|u^k - u^*\|_M^2 - \frac{1}{2}\|u^{k+1} - u^k\|_M^2. \end{aligned}$$

- Next we shall assume R is “strongly monotone”.

Linear convergence under strong monotonicity

Strongly monotone operator

- R is said μ-**strongly monotone** if $R - \mu I$ is monotone.
- For proper, convex, lsc function J, ∂J is μ-strongly monotone iff J is μ-strongly convex, i.e., $J - \frac{\mu}{2}\|\cdot\|^2$ is convex.

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- R is μ-strongly monotone

$$\Rightarrow \left\langle u^{k+1} - u^*, \xi^{k+1} - 0 \right\rangle \geq \mu \|u^{k+1} - u^*\|^2$$

$$\Rightarrow \left(\frac{1}{2} + \frac{\mu}{\lambda_{\max}(M)}\right) \|u^{k+1} - u^*\|_M^2$$

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- Recall in PDHG:

$$R = \begin{bmatrix} \partial G & K^\top \\ -K & \partial F^* \end{bmatrix}.$$

R is μ-strongly monotone $\Leftrightarrow$ G, F^* are μ-strongly convex; F^* is μ-strongly convex $\Leftrightarrow$ F is $\frac{1}{\mu}$-Lipschitz differentiable.