

Proof Script for WS18–19 Convex Optimization*

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1 Convex Analysis

Theorem 1.1 (separation of convex sets). *Let C_1, C_2 be nonempty convex subsets in \mathbb{E} .*

1. *Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

2. *Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

3. *Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

4. *Assume $\emptyset \neq \text{int } C_1 \subset \mathbb{E} \setminus C_2$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

Proof. (1) Consider the projection of w onto C_1 , i.e., set $u^* := \arg \min_{u \in C_1} \frac{1}{2} \|u - w\|^2$ or, equivalently via the variational inequality: $\langle u - u^*, u^* - w \rangle \geq 0 \forall u \in C_1$. Now let $v := w - u^* \neq 0$. Then $\forall u \in C_1$, we have $\langle v, w \rangle = \langle w - u^*, w \rangle = \|w - u^*\|^2 + \langle w - u^*, u^* \rangle \geq \|w - u^*\|^2 + \langle w - u^*, u \rangle = \|v\|^2 + \langle v, u \rangle$. Set $\alpha := \sup\{\langle v, u \rangle : u \in C_1\}$. Note $\alpha < \infty$ since $\langle v, u \rangle \leq \langle v, u^* \rangle \forall u \in C$. Thus $\langle v, w \rangle > \alpha \geq \langle v, u \rangle \forall u \in C_1$.

(2) Since $\mathbb{E} \setminus C_1$ is closed, $\exists w^k \in \mathbb{E} \setminus \text{cl } C_1$ s.t. $w^k \rightarrow w$. For each w^k , by (i), $\exists v^k \in \mathbb{E}$ with $\|v^k\| \equiv 1$ s.t. $\langle v^k, w^k \rangle \leq \langle v^k, u^1 \rangle \forall u^1 \in C_1 \subset \text{cl } C_1$. Hence $v^k \rightarrow v \in \mathbb{E}$ along a subsequence s.t. $\|v\| = 1$ and $\alpha := \langle v, w \rangle \leq \langle v, u \rangle \forall u \in C_1$.

(3) Let $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1, u^2 \in C_2\}$. Note that C is a convex, open set, and $0 \notin C$. By (2), $\exists v \in \mathbb{E}$ with $\|v\| = 1$ s.t. $\langle -v, u^2 - u^1 \rangle \geq \langle -v, 0 \rangle = 0$ or, equivalently, $\langle v, u^1 \rangle \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$. Set $\alpha := \sup\{\langle v, u^2 \rangle : u^2 \in C_2\}$, then we conclude that $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$.

(4) By applying (3) to $\text{int } C_1$ and C_2 , we have $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in \text{int } C_1, u^2 \in C_2$. The inequality remains true for all $u_1 \in C_1$. \square

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