

Proof Script for WS2018/19 Convex Optimization*

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1 Convex Analysis

Theorem 1.1 (separation of convex sets). *Let C_1, C_2 be nonempty convex subsets in \mathbb{E} .*

1. *Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

2. *Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

3. *Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

4. *Assume $\emptyset \neq \text{int } C_1 \subset \mathbb{E} \setminus C_2$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

Proof. (1) Consider the projection of w onto C_1 , i.e., set $u^* := \arg \min_{u \in C_1} \frac{1}{2} \|u - w\|^2$ or, equivalently via the variational inequality: $\langle u - u^*, u^* - w \rangle \geq 0 \forall u \in C_1$. Now let $v := w - u^* \neq 0$. Then $\forall u \in C_1$, we have $\langle v, w \rangle = \langle w - u^*, w \rangle = \|w - u^*\|^2 + \langle w - u^*, u^* \rangle \geq \|w - u^*\|^2 + \langle w - u^*, u \rangle = \|v\|^2 + \langle v, u \rangle$. Set $\alpha := \sup\{\langle v, u \rangle : u \in C_1\}$. Note $\alpha < \infty$ since $\langle v, u \rangle \leq \langle v, u^* \rangle \forall u \in C$. Thus $\langle v, w \rangle > \alpha \geq \langle v, u \rangle \forall u \in C_1$.

(2) Since $\mathbb{E} \setminus C_1$ is closed, $\exists w^k \in \mathbb{E} \setminus \text{cl } C_1$ s.t. $w^k \rightarrow w$. For each w^k , by (i), $\exists v^k \in \mathbb{E}$ with $\|v^k\| \equiv 1$ s.t. $\langle v^k, w^k \rangle \leq \langle v^k, u \rangle \forall u \in \text{cl } C_1$. Hence $v^k \rightarrow v \in \mathbb{E}$ along a subsequence s.t. $\|v\| = 1$ and $\alpha := \langle v, w \rangle \leq \langle v, u \rangle \forall u \in C_1 \subset \text{cl } C_1$.

(3) Let $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1, u^2 \in C_2\}$. Note that C is a convex, open set, and $0 \notin C$. By (2), $\exists v \in \mathbb{E}$ with $\|v\| = 1$ s.t. $\langle -v, u^2 - u^1 \rangle \geq \langle -v, 0 \rangle = 0$ or, equivalently, $\langle v, u^1 \rangle \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$. Set $\alpha := \sup\{\langle v, u^2 \rangle : u^2 \in C_2\}$, then we conclude that $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$.

(4) By applying (3) to $\text{int } C_1$ and C_2 , we have $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in \text{int } C_1, u^2 \in C_2$. The inequality remains true for all $u_1 \in C_1$. \square

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Theorem 1.2. *A proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \text{rint dom } J$.*

Proof. Throughout the proof, we consider $J : \text{aff dom } J \rightarrow \overline{\mathbb{R}}$.

(i) Claim: If $M = \sup\{J(v) : v \in B_\epsilon(u)\} < \infty$ with $\epsilon > 0$, then J is locally Lipschitz at u .

First, by convexity of J we have $\forall v \in B_\epsilon(u) : J(v) \geq 2J(u) - J(2u - v) \geq 2J(u) - M$. Thus, $\sup\{|J(v)| : v \in B_\epsilon(u)\} \leq M + 2|J(u)|$.

Next, we show J is Lipschitz on $B_{\epsilon/2}(u)$. Let $v, w \in B_{\epsilon/2}(u)$ be given. Take $z \in B_\epsilon(u)$ s.t. $w = (1 - t)v + tz$ for some $t \in [0, 1]$ and $\|z - v\| \geq \epsilon/2$. By convexity, $J(w) - J(v) \leq t(J(z) - J(v)) \leq 2t(M - J(u))$. Since $t(z - v) = w - v$, we have $t = \|w - v\|/\|z - v\| \leq 2\|w - v\|/\epsilon$ and $J(w) - J(v) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Analogously, one can show $J(v) - J(w) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Hence, J is Lipschitz on $B_{\epsilon/2}(u)$ with modulus $4(M - J(u))/\epsilon$.

(ii) Let $u \in \text{rint dom } J$ and $n = \dim(\text{aff dom } J)$. Then by Carathéodory's theorem, $\exists\{\alpha^i\}_{i=1}^{n+1} \subset (0, 1)$, $\{u^i\}_{i=1}^{n+1} \subset \text{dom } J$ s.t. $u = \sum_{i=1}^{n+1} \alpha^i u^i$, $\sum_{i=1}^{n+1} \alpha^i = 1$, i.e., u belongs to the interior of the convex hull of $\{u^i\}_{i=1}^{n+1}$. Thus one can apply (i) to assert that J is locally Lipschitz at u . \square

Theorem 1.3. *For any proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, if $u^* \in \text{dom } J$ is a local minimizer of J , then it is also a global minimizer.*

Proof. By the definition of a local minimizer, $\exists\epsilon > 0$ s.t. $J(u^*) \leq J(u) \forall u \in B_\epsilon(u^*)$. For the sake of contradiction, assume $\exists\bar{u} \in \mathbb{E}$ s.t. $J(\bar{u}) < J(u^*)$. By convexity of J , we have $J(\alpha\bar{u} + (1 - \alpha)u^*) \leq J(u^*) - \alpha(J(u^*) - J(\bar{u})) < J(u^*) \forall \alpha \in (0, 1]$. This violates the local optimality of u^* as $\alpha \rightarrow 0^+$. \square

Theorem 1.4. *Any proper function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc, has a (global) minimizer.*

Proof. Let $\{u^k\}$ be an infimizing sequence for J , i.e., $\lim_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u) > -\infty$. Since $\{J(u^k)\}$ is uniformly bounded from above, by coercivity of J $\{u^k\}$ is uniformly bounded. By compactness, $u^k \rightarrow u^*$ along a subsequence. Since J is lsc, we have $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u)$, which implies $J(u^*) = \inf_{u \in \mathbb{E}} J(u)$ or u^* is a minimizer of J . \square

Theorem 1.5. *The minimizer of a strictly convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is unique.*

Proof. Let $u, v \in \mathbb{E}$ be two (global) minimizers s.t. $u \neq v$ and $J(u) = J(v) = J^*$. By strict convexity of J , $J(\alpha u + (1 - \alpha)v) < \alpha J(u) + (1 - \alpha)J(v) = J^*$ for all $\alpha \in (0, 1)$, which contradicts the global optimality of u and v . \square

Theorem 1.6. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \text{int dom } J$, $\partial J(u)$ is a nonempty, compact, and convex subset.*

Proof. (i) nonemptiness. Since $(u, J(u)) \notin \text{int epi } J$, by Theorem 1.1 we have $\exists(p, -\alpha) \in \mathbb{E} \times \mathbb{R}$ s.t. $(p, -\alpha) \neq (0, 0)$, $\alpha \geq 0$ by our choice, and $\langle (p, -\alpha), (u - v, J(u) - J(v)) \rangle \geq 0 \forall v \in \text{dom } J$. In fact, we must have $\alpha > 0$ since otherwise $p = 0$. Thus, we conclude that $p/\alpha \in \partial J(u)$.

(ii) boundedness. By Theorem 1.2, J is locally Lipschitz at u with modulus L_u . Let $p \in \partial J(u)$ be fixed. For any $h \in (\text{dom } J) - u$ whenever $\|h\|$ is sufficiently small, we have $\langle p, h \rangle \leq J(u + h) - J(u) \leq L_u\|h\|$. This holds true only if $\|p\| \leq L_u$, which implies boundedness of $\partial J(u)$.

(iii) closedness. Let $v \in \mathbb{E}$ be arbitrarily fixed and $p^k \rightarrow p^*$ where each $p^k \in \partial J(u)$. Then $\forall k : J(v) - J(u) \geq \langle p^k, v - u \rangle$. By continuity, $J(v) - J(u) \geq \langle p^*, v - u \rangle$ when passing $k \rightarrow \infty$. Since v can be arbitrary, we assert $p^* \in \partial J(u)$.

(iv) convexity. Let $v \in \mathbb{E}$ be arbitrarily fixed, and $p, q \in \partial J(u)$. Then we have

$$\begin{aligned} J(v) &\geq J(u) + \langle p, v - u \rangle, \\ J(v) &\geq J(u) + \langle q, v - u \rangle. \end{aligned}$$

Hence, $\forall 0 \leq \alpha \leq 1 : J(v) \geq J(u) + \langle \alpha p + (1 - \alpha)q, v - u \rangle$, i.e., $\alpha p + (1 - \alpha)q \in \partial J(u)$. \square

Theorem 1.7. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a convex function. Then ∂J is a monotone operator, i.e. $\forall u^1, u^2 \in \text{dom } J, p^1 \in \partial J(u^1), p^2 \in \partial J(u^2) :$*

$$\langle p^1 - p^2, u^1 - u^2 \rangle \geq 0.$$

Proof. By applying the definition of subdifferential at arbitrarily given $u^1, u^2 \in \text{dom } J$, we have

$$\begin{aligned} J(u^2) &\geq J(u^1) + \langle p^1, u^2 - u^1 \rangle, \\ J(u^1) &\geq J(u^2) + \langle p^2, u^1 - u^2 \rangle. \end{aligned}$$

Adding the two inequalities yields $\langle p^1 - p^2, u^1 - u^2 \rangle \geq 0$. \square

Theorem 1.8. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then ∂J is a closed set-valued map, i.e., $p^* \in \partial J(u^*)$ whenever*

$$\exists (u^k, p^k) \rightarrow (u^*, p^*) \in (\text{dom } J) \times \mathbb{E} \text{ s.t. } p^k \in \partial J(u^k) \forall k.$$

Proof. Let $v \in \mathbb{E}$ be arbitrarily fixed. For each $k, p^k \in \partial J(u^k) \Rightarrow J(v) \geq J(u^k) + \langle p^k, v - u^k \rangle$. Passing $k \rightarrow \infty$, we have $\langle p^k, v - u^k \rangle \rightarrow \langle p^*, v - u^* \rangle$ and $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k)$. Hence, $J(u^*) + \langle p^*, v - u^* \rangle \leq \liminf_{k \rightarrow \infty} \{J(u^k) + \langle p^k, v - u^k \rangle\} \leq J(v)$. Since v can be arbitrary, $p^* \in \partial J(u^*)$. \square

Theorem 1.9. *Given any proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is: $0 \in \partial J(u^*)$.*

Proof. (i) sufficiency. $0 \in \partial J(u^*) \Rightarrow J(u) \geq J(u^*) + \langle 0, u - u^* \rangle = J(u^*) \forall u \in \mathbb{E}$.

(ii) necessity. $J(u^*) \leq J(u) \forall u \in \mathbb{E} \Rightarrow J(u^*) + \langle 0, u - u^* \rangle \leq J(u) \forall u \Rightarrow 0 \in \partial J(u^*)$. \square