Convex Optimization for Machine Learning and Computer Vision

Lecture: Dr. Tao Wu
Exercises: Yuesong Shen, Zhenzhang Ye
Winter Semester 2018/19

Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 3

Room: 01.09.014

Wednesday, 14.11.2018, 12:15-14:00

Submission deadline: Monday, 12.11.2018, 16:15, Room 01.09.014

Subdifferential (14+6 Points)

Exercise 1 (4 Points)**.** Let the convex function $J : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be differentiable at $u \in \text{int}(\text{dom}(J))$. Show that

$$\partial J(u) = \{\nabla J(u)\}.$$

Hint: Use the definition of the subdifferential and the directional derivative. For J being differentiable at the interior of its domain, some direction $v \in \mathbb{R}^n$ and some point $u \in \text{int}(\text{dom}(J))$ the directional derivative $\partial_v J$ of J is given as

$$\partial_v J(u) := \lim_{\epsilon \to 0} \frac{J(u + \epsilon v) - J(u)}{\epsilon} = \lim_{\epsilon \to 0} \frac{J(u) - J(u - \epsilon v)}{\epsilon} = \langle \nabla J(u), v \rangle.$$

Exercise 2 (6 Points)**.** Compute the subdifferential of norms in Euclidean space:

- Let $\|\cdot\|$ be a norm on an Euclidean space $\mathbb{E}$, and $\|\cdot\|_*$ its dual norm defined as

  $$\|p\|_* = \sup_{\|x\| \le 1} \langle p, x \rangle,$$

  prove that

  $$\partial \|\cdot\| (x) = \{p \in \mathbb{E} : \langle p, x \rangle = \|x\|, \|p\|_* \le 1\}. \quad (1)$$

  Hint: For $x \neq 0$, we have a generalized Cauchy-Schwarz inequality:

  $$\langle x, y \rangle = \|x\| \langle \frac{x}{\|x\|}, y \rangle \le \|x\| \cdot \sup_{\|z\| \le 1} \langle z, y \rangle = \|x\| \|y\|_*, \ \forall x, y \in \mathbb{E}. \quad (2)$$

- Using the result above, compute the subdifferential of the following functions:
  - $J : \mathbb{R}^n \to \mathbb{R}, J(u) = \|u\|_1$.
  - $J : \mathbb{R}^n \to \mathbb{R}, J(u) = \|u\|_2$.
  - $J : \mathbb{R}^n \to \mathbb{R}, J(u) = \|u\|_\infty$.

Exercise 3 (4 points)**.** Given $b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$ with linearly independent rows, show that the normal cone N_C of the linear-inequality constraints

$$C = \{u \in \mathbb{R}^n : Au \leq b, \} \tag{3}$$

is

$$N_C(u) = \{A^\top \lambda : \lambda \geq 0, \lambda_i = 0 \text{ if } (Au - b)_i < 0\}. \tag{4}$$

Exercise 4 (6 points)**.** Compute the subdifferential of nuclear norm:

$$X \in \mathbb{R}^{n\times n} \mapsto \|X\|_{nuclear} = \sum_i \sigma_i(X),$$

i.e., sum of singular values.
Hint: Show that the subdifferential at point $X \in \mathbb{R}^{n\times n}$ with $s \geq 0$ zero singular values is given as

$$\partial \left\|\cdot\right\|_{\text{nuc}}(X) = \left\{U_1 V_1^\top + U_2 M V_2^\top : M \in \mathbb{R}^{s\times s}, \|M\|_{\text{spec}} \leq 1\right\}, \tag{5}$$

where $U = \begin{bmatrix}U_1 & U_2\end{bmatrix}$ and $V = \begin{bmatrix}V_1 & V_2\end{bmatrix}$ are given by the singular value decomposition of $X = U\Sigma V^\top$, with U_1 and V_1 having $n - s$ columns. Furthermore $\|\cdot\|_{\text{spec}}$ denotes the spectral norm, i.e., the largest singular value.