Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 4

Room: 01.09.014

Wednesday, 21.11.2018, 12:15-14:00

Submission deadline: Monday, 19.11.2018, 16:15, Room 01.09.014

Convex conjugate (14+6 Points)

Exercise 1 (4 points)**.** Let $A \in \mathbb{R}^{n\times n}$ be orthonormal, meaning that $A^\top A = AA^\top = I$. Let the convex set C be given as

$$C := \{u \in \mathbb{R}^n : \|Au\|_\infty \leq 1\}.$$

Compute a formula for the projection onto C given as

$$\Pi_C(v) := \text{argmin}_{u\in\mathbb{R}^n} \frac{1}{2}\|u - v\|_2^2, \quad \text{s.t. } u \in C.$$

Hint: Show that the ℓ_2-norm of a vector is invariant under a multiplication with an orthonormal matrix A, meaning that $\|u\|_2 = \|Au\|_2$.

Exercise 2 (6 points)**.** Assume $J : \mathbb{R}^n \to \mathbb{R}$, compute the convex conjugate of following functions:

- $J(u) = \frac{1}{q}||u||_q^q = \sum_{i=1}^n \frac{1}{q}|u_i|^q$, $q \in [1, +\infty]$.
- $J(u) = \sum_{i=1}^n u_i \text{log} u_i + \delta_{\triangle^{n-1}}(u)$.
- $J(u) = \begin{cases} \frac{1}{2}\|u\|_2^2, & \|u\|_2 \leq \epsilon \\ +\infty, & \text{otherwise} \end{cases}$

Exercise 3 (4 points)**.** Assume $J : \mathbb{E} \to \mathbb{R}$, prove following facts of convex conjugate:

- $\tilde{J}(\cdot) = \alpha J(\cdot) \Rightarrow \tilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$, $\alpha > 0$.
- $\tilde{J}(\cdot) = J(\cdot - z) \Rightarrow \tilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z\rangle$.

Exercise 4 (6 points)**.** Show that projection onto a convex set is Lipschitz continuous with constant equals 1, i.e.

$$||\Pi_C(u) - \Pi_C(v)|| \leq ||u - v||, \ \forall u, v \in \mathbb{E}$$

where C is a convex set.