

Weekly Exercises 5

Room: 01.09.014

Wednesday, 28.11.2018, 12:15-14:00

Submission deadline: Monday, 26.11.2018, 16:15, Room 01.09.014

Convex conjugate and prox (8+6 Points)

Exercise 1 (4 points). Given a $X \in \mathbb{R}^{m \times n}$, compute the subdifferential of the 1,2-norm, i.e.

$$\partial \|X\|_{1,2} = \partial \left(\sum_{i=1}^m \left(\sum_{j=1}^n X_{i,j}^2 \right)^{1/2} \right).$$

Exercise 2 (4 Points). Consider following problems of convex conjugate:

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Show that the convex conjugate of the perspective function $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$

$$g(x, t) = \begin{cases} tf(x/t), & \text{if } t > 0 \\ \infty, & \text{otherwise} \end{cases}$$

is given by

$$g^*(y, s) = \begin{cases} 0, & \text{if } f^*(y) \leq -s \\ \infty, & \text{otherwise} \end{cases}$$

- Show that the biconjugate of the perspective function g is given by

$$g^{**}(x, t) = \begin{cases} tf(x/t), & \text{if } t > 0 \\ \sigma_{\text{dom}(f^*)}(x), & \text{if } t = 0 \\ \infty, & \text{if } t < 0 \end{cases}$$

where $\sigma_{\text{dom}(f^*)}(x) = \sup_{y \in \text{dom}(f^*)} \langle x, y \rangle$ is the *support function* of $\text{dom}(f^*)$.

Exercise 3 (6 Points). Let $A \in \mathbb{R}^{m \times n}$ be a linear operator and $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ a convex function. Then $Af : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ defined as

$$(Af)(u) := \begin{cases} \inf_{v \in \mathbb{R}^n, Av=u} f(v) & \text{if } \exists v \in \mathbb{R}^n \text{ s.t. } Av = u \\ \infty & \text{otherwise.} \end{cases}$$

is called the image of f under A .

1. Show that the convex conjugate $(Af)^*$ of Af is given as $f^* \circ A^\top$ where $(f^* \circ A^\top)(v) := f^*(A^\top v)$.
2. Name the properties that we require for $A^\top f^* = (f \circ A)^*$ to hold. What theorem from the lecture applies here?
3. Give an example of a closed, convex and non-empty set C and a linear operator A s.t. $AC := \{Ax : x \in C\}$ is not closed.
4. Let f be closed, (convex) and proper. Argue that Af does not need to be closed.