

Weekly Exercises 12

Room: 01.09.014

Wednesday, 30.01.2019, 12:15-14:00

Submission deadline: Monday, 28.01.2019, 16:15, Room 01.09.014

Consensus optimization (4+4 Points)

Exercise 1 (8 Points). Consider the *consensus optimization* problem:

$$\begin{aligned} \min_{\{x_i\}_{i=1}^l \subset \mathbb{R}^n, x_0 \in \mathbb{R}^n} \quad & \sum_{i=1}^l f_i(x_i) \\ \text{subject to} \quad & x_i = x_0 \quad \forall i \in \{1, 2, \dots, l\}. \end{aligned} \tag{1}$$

Here each function $f_i : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ is proper, convex, and lower-semicontinuous.

- Write down the augmented Lagrangian functional for (1) (which will involve multipliers $\{y_i\}_{i=1}^l \subset \mathbb{R}^n$).
- Formulate an alternating direction of multipliers (ADMM) method for (1). Update the variables in the order of $\{x_i\}_{i=1}^l, \{y_i\}_{i=1}^l, x_0$.
- One can interpret the ADMM scheme in (b) as a generalized proximal iteration on $(x_0, \{y_i\}_{i=1}^l)$:

$$0 \in M \begin{bmatrix} x_0^{k+1} - x_0^k \\ y_1^{k+1} - y_1^k \\ \vdots \\ y_l^{k+1} - y_l^k \end{bmatrix} + R \left(\begin{bmatrix} x_0^{k+1} \\ y_1^{k+1} \\ \vdots \\ y_l^{k+1} \end{bmatrix} \right).$$

Identify the positive semidefinite matrix M and the monotone operator R in the above equation.

Solution. • Augmented Lagrangian is defined as:

$$\mathcal{L}_\tau(x_0, \{x_i\}_{i=1}^l, \{y_i\}_{i=1}^l) = \sum_{i=1}^l \left(f_i(x_i) - \langle y_i, x_i - x_0 \rangle + \frac{\tau}{2} \|x_i - x_0\|_2^2 \right)$$

with $\tau > 0$.

- ADMM can be formulated as:

$$\begin{aligned} x_i^{k+1} &= \arg \min_{x_i} f_i(x_i) - \langle y_i^k, x_i \rangle + \frac{\tau}{2} \|x_i - x_0^k\|_2^2 \\ &= (\partial f_i + \tau I)^{-1}(\tau x_0^k + y_i^k) \quad \forall i \in \{1, \dots, l\}, \end{aligned} \quad (2)$$

$$y_i^{k+1} = y_i^k - \tau(x_i^{k+1} - x_0^k) \quad \forall i \in \{1, \dots, l\}, \quad (3)$$

$$\begin{aligned} x_0^{k+1} &= \arg \min_{x_0} \sum_{i=1}^l \left(\langle y_i^{k+1}, x_0 \rangle + \frac{\tau}{2} \|x_i^{k+1} - x_0\|_2^2 \right) \\ &= \frac{1}{l} \sum_{i=1}^l \left(x_i^{k+1} - \frac{1}{\tau} y_i^{k+1} \right). \end{aligned} \quad (4)$$

- (2), (3) $\Rightarrow \forall i : y_i^{k+1} \in \partial f_i(x_i^{k+1}) \Leftrightarrow x_i^{k+1} \in \partial f_i^*(y_i^{k+1})$. By eliminating $\{x_i^{k+1}\}$, we have

$$\begin{aligned} 0 &\in \partial f_i^*(y_i^{k+1}) - x_0^k + \frac{1}{\tau}(y_i^{k+1} - y_i^k), \\ 0 &= \tau l(x_0^{k+1} - x_0^k) + \sum_{i=1}^l (2y_i^{k+1} - y_i^k), \end{aligned}$$

or equivalently

$$0 \in M \begin{bmatrix} x_0^{k+1} - x_0^k \\ y_1^{k+1} - y_1^k \\ \vdots \\ y_l^{k+1} - y_l^k \end{bmatrix} + R \left(\begin{bmatrix} x_0^{k+1} \\ y_1^{k+1} \\ \vdots \\ y_l^{k+1} \end{bmatrix} \right),$$

where

$$M = \begin{bmatrix} \tau l I & I & \dots & I \\ I & \frac{1}{\tau} I & & 0 \\ \vdots & & \ddots & \\ I & 0 & & \frac{1}{\tau} I \end{bmatrix}, \quad R = \begin{bmatrix} 0 & I & \dots & I \\ -I & \partial f_1^* & & 0 \\ \vdots & & \ddots & \\ -I & 0 & & \partial f_l^* \end{bmatrix}$$