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Weekly Exercises 2

Room: 01.09.014

Wednesday, 07.11.2018, 12:15-14:00

Submission deadline: Monday, 05.11.2018, 16:15, Room 01.09.014

Theory: Convex Functions (10+8 Points)

Exercise 1 (4 Points)**.** Let $J : \mathbb{E} \to \bar{\mathbb{R}}$ be proper. Prove the equivalence of the following statements:

- J is lower semi-continuous (l.s.c).
- The epigraph of J is closed.

Solution. "$\Rightarrow$": Assume J is $l.s.c$, we want to show the epigraph is closed. According to the exercise in sheet 1, we only need to show all the sequences attains their limit points inside the epigraph.
For any sequence (u_n, α_n) in epi J converging to certain (u^*, α^*), we show that (u^*, α^*) is in epi J as well. Using the definition of $l.s.c$, we have :

$$\begin{aligned} J(u^*) &= \liminf_{u \to u^*} J(u) \\ &\leq \liminf_{n \to \infty} J(u_n) \\ &\leq \liminf_{n \to \infty} \alpha_n \\ &= \alpha^*. \end{aligned}$$

where the first inequality comes from the definition of $\liminf$ and the second one due to (u_n, α_n) in epi J. Therefore, (u^*, α^*) is in epi J as well.
"$\Leftarrow$": Assume J is not l.s.c, we try to construct a sequence in epi J but attains its limit points not in epi J.
Since J is not l.s.c, we have:

$$\exists u \in \text{dom } J,\ \exists \{u_n\}_{n \in \mathbb{N}} \to u,\ \text{s.t. } J(u) > \liminf_{n \to \infty} J(u_n) \tag{1}$$

Therefore, we can find a N and $\epsilon > 0$, such that $\forall n \geq N,\ J(u) - \epsilon \geq J(u_n)$. This states that we find a sequence $(u_n, J(u) - \epsilon)$ which are in epi J. But its limitation $(u, J(u) - \epsilon)$ is not in epi J.

Exercise 2 (4 Points)**.** Suppose $J : \mathbb{E} \to \mathbb{R}$ is convex with dom$J = \mathbb{R}^n$, and bouned above on $\mathbb{R}^n$. Show that J is a constant function.

Solution. Assume that J is not a constant function, we then can pick $u \neq v$ such that $J(u) > J(v)$. According to the convexity of J, we have:

$$J(u) \leq \alpha J(\frac{u-(1-\alpha)v}{\alpha}) + (1-\alpha)J(v), \ \forall \alpha \in (0,1)$$

Dividing α on both sides:

$$\begin{aligned} \frac{J(u)-(1-\alpha)J(v)}{\alpha} &\leq J(\frac{u-(1-\alpha)v}{\alpha}) \\ \Rightarrow \frac{J(u)-J(v)}{\alpha} + J(v) &\leq J(\frac{u-(1-\alpha)v}{\alpha}) \end{aligned}$$

Since $J(u) > J(v)$, when $\alpha \to 0$, left side goes to $+\infty$. Therefore, J is not bounded above. It is shown by contradiction.

Exercise 3 (6 Points)**.** Show that the following functions $J : \mathbb{E} \to \overline{\mathbb{R}}$ are convex:

- $J(u) = \|u\|$, for any norm $\|\cdot\|$ over a normed vector space.
- $J(u) = F(Ku)$, for convex $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ and linear $K : \mathbb{R}^m \to \mathbb{R}^n$.
- $J(u) = \max\{J_1(u), J_2(u)\}$, where J_1 and J_2 are convex functions with $\mathbb{E} \to \overline{\mathbb{R}}$.

Solution.

- Take $u, v \in \mathbb{R}^n$, $\lambda \in [0,1]$:

$$\begin{aligned} J(\lambda u + (1-\lambda)v) = \|\lambda u + (1-\lambda)v\| \leq \\ \|\lambda u\| + \|(1-\lambda)v\| = \lambda \|u\| + (1-\lambda)\|v\|. \end{aligned} \tag{2}$$

- Take $u, v \in \operatorname{dom} J$, $\lambda \in [0,1]$.

$$\begin{aligned} J(\lambda u + (1-\lambda)v) := F(K(\lambda u + (1-\lambda)v) = \\ F(\lambda Ku + (1-\lambda)Kv)) \leq \\ \lambda F(Ku) + (1-\lambda)F(Kv) = \underbrace{\lambda J(u) + (1-\lambda)J(v)}_{<\infty, \text{ since } u,v \in \operatorname{dom} J} \end{aligned} \tag{3}$$

  This shows that J is convex on its domain and $\operatorname{dom} J$ is a convex set.

- It is easy to see that dom J = dom $J_1 \cap$dom J_2. Intersection of two convex sets are still convex.
  Then take $u, v \in \operatorname{dom} J$, $\lambda \in [0,1]$. Without loss of generality, we assume $J_1 \geq J_2$ at $\lambda u + (1-\lambda)v$:

$$\begin{aligned} J(\lambda u + (1-\lambda)v) &= \max\{J_1(\lambda u + (1-\lambda)v), J_2(\lambda u + (1-\lambda)v)\} \\ &= J_1(\lambda u + (1-\lambda)v) \\ &\leq \lambda J_1(u) + (1-\lambda)J_1(v) \\ &\leq \lambda \max\{J_1(u), J_2(u)\} + (1-\lambda)\max\{J_1(v), J_2(v)\} \end{aligned} \tag{4}$$

Exercise 4 (4 Points)**.** Let $U \subset \mathbb{E}$ open and convex and let $J : U \to \mathbb{R}$ be twice continuously differentiable. Prove the equivalence of the following statements:

- J is convex.
- For all $u \in U$ the Hessian $\nabla^2 J(u)$ is positive semidefinite ($\forall\, v \in \mathbb{E} : v^\top \nabla^2 J(u) v \geq 0$).

Hints: You can use that for $u, v \in U$ it holds that J is convex iff

$$(v - u)^\top \nabla J(u) \leq J(v) - J(u).$$

Further recall that there are two variants of the Taylor expansion:

$$J(u + td) = J(u) + td^\top \nabla J(u) + \frac{t^2}{2} d^\top \nabla^2 J(u) d + o(t^2)$$

with $\lim_{t \to 0} \frac{o(t^2)}{t^2} = 0$ and

$$J(u + d) = J(u) + d^\top \nabla J(u) + \frac{1}{2} d^\top \nabla^2 J(u + td) d$$

for appropriate $t \in (0, 1)$.

Solution. Let J be convex, $u \in U$ and $d \in \mathbb{R}^n$. Since U is open there exists $\tau > 0$ s.t. for all $t \in (0, \tau]$ we have that $u + td \in U$. Using the Taylor expansion given in the hint we obtain

$$0 \overset{\text{Hint}}{\leq} J(u + td) - J(u) - td^\top \nabla J(u) = \frac{t^2}{2} d^\top \nabla^2 J(u) d + o(t^2)$$

Multiplying both sides with $\frac{2}{t^2}$ yields

$$0 \leq d^\top \nabla^2 J(u) d + 2 \underbrace{\frac{o(t^2)}{t^2}}_{\to 0}.$$

Let conversely $\nabla^2 J(z)$ be positive semidefinite for all $z \in U$ and let $u, v \in U$. Using the Taylor expansion we have

$$J(v) = J(u + (v - u)) = J(u) + (v - u)^\top \nabla J(u) + \frac{1}{2} \underbrace{(v - u)^\top \nabla^2 J(u + t(v - u))(v - u)}_{\geq 0 \text{ by assumption.}}$$

for certain $t \in (0, 1)$ and therefore

$$J(v) - J(u) \geq (v - u)^\top \nabla J(u),$$

which means that J is convex.

Programming: Inpainting(Due date: 12.11) (12 Points)

Exercise 5 (12 Points)**.** Write a program in MATLAB (or Python) that solves the inpainting problem for the vegetable image:

$$\min_{u \in \mathbb{R}^{n \times m}} \sum_{i,j} (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t. } u_{i,j} = f_{i,j} \ \forall (i,j) \in I,$$

with index set I of pixels to keep. Those can be identified as the white pixels of the mask image.
Hint: The constrained optimization problem can be reformulated so that it becomes unconstrained: Rewrite the objective as a least squares problem in terms of the unknown intensities $u_{i,j}$, $(i,j) \notin I$ using sparse linear operators: Find linear operators X, Y s.t. u can be decomposed as

$$u = X\tilde{u} + Yf$$

where $\tilde{u}$ contains only the unknown intensities. Optimize for $\tilde{u}$ instead of u. You may use MATALBs `mldivide` (for Python, check out e.g. `scipy.sparse.linalg.{spsolve, lsqr}`).