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Weekly Exercises 4

Room: 01.09.014

Wednesday, 21.11.2018, 12:15-14:00

Submission deadline: Monday, 19.11.2018, 16:15, Room 01.09.014

Convex conjugate (14+6 Points)

Exercise 1 (4 points)**.** Let $A \in \mathbb{R}^{n\times n}$ be orthonormal, meaning that $A^\top A = AA^\top = I$. Let the convex set C be given as

$$C := \{u \in \mathbb{R}^n : \|Au\|_\infty \le 1\}.$$

Compute a formula for the projection onto C given as

$$\Pi_C(v) := \text{argmin}_{u\in\mathbb{R}^n} \frac{1}{2}\|u-v\|_2^2, \quad \text{s.t. } u \in C.$$

Hint: Show that the ℓ_2-norm of a vector is invariant under a multiplication with an orthonormal matrix A, meaning that $\|u\|_2 = \|Au\|_2$.

Solution. We begin proving the hint:

$$\|Ax\|_2^2 = \langle Ax, Ax\rangle = \langle A^\top Ax, x\rangle = \langle x, x\rangle = \|x\|_2^2.$$

The idea is to rewrite the projection onto the set C in terms of the projection $\Pi_{\tilde{C}}$ onto the unit ball of the ℓ_∞-norm $\tilde{C} := \{x \in \mathbb{R}^n : \|x\|_\infty \le 1\}$. With the substitution

$$w := Au \iff u = A^\top w$$

and using the hint we obtain:

$$\begin{aligned}\Pi_C(v) &= \text{argmin}_{\|Au\|_\infty\le 1} \frac{1}{2}\|v-u\|^2\\ &= A^\top \text{argmin}_{\|w\|_\infty\le 1} \frac{1}{2}\|v-A^\top w\|^2\\ &= A^\top \text{argmin}_{\|w\|_\infty\le 1} \frac{1}{2}\|A(v-A^\top w)\|^2\\ &= A^\top \text{argmin}_{\|w\|_\infty\le 1} \frac{1}{2}\|Av-AA^\top w\|^2\\ &= A^\top \text{argmin}_{\|w\|_\infty\le 1} \frac{1}{2}\|Av-w\|^2\\ &= A^\top \Pi_{\tilde{C}}(Av).\end{aligned}$$

Exercise 2 (6 points)**.** Assume $J : \mathbb{R}^n \to \mathbb{R}$, compute the convex conjugate of following functions:

- $J(u) = \frac{1}{q}||u||_q^q = \sum_{i=1}^n \frac{1}{q}|u_i|^q$, $q \in [1, +\infty]$.
- $J(u) = \sum_{i=1}^n u_i \text{log} u_i + \delta_{\triangle^{n-1}}(u)$.
- $J(u) = \begin{cases} \frac{1}{2}\|u\|_2^2, & \|u\|_2 \leq \epsilon \\ +\infty, & \text{otherwise} \end{cases}$

Solution.

- $J^*(v) = \sup_u \langle u, v\rangle - J(u)$. Since it is separable, we apply first-order optimality condition elementwisely:

$$\sup_{u_i} \langle u_i, v_i\rangle - \frac{1}{q}(|u_i|)^q \Rightarrow 0 = v_i - |u_i|^{q-1}\text{sign}(u_i) \Rightarrow u_i = |v_i|^{1/(q-1)}\text{sign}(v_i)$$

Substitute u_i back to the first equation, we have

$$\begin{aligned} J^*(v)_i &= |v_i|^{q/(q-1)} - \frac{1}{q}|v_i|^{q/(q-1)} \\ &= (1 - \frac{1}{q})|v_i|^{q/(q-1)} \\ &= (1 - \frac{1}{q})|v_i|^{1/(1-\frac{1}{q})} \end{aligned}$$

Substituting $\frac{1}{p} = 1 - \frac{1}{q}$, we get $J^*(v) = \frac{1}{p}||v||_p^p$.

- Consider the convex conjugate elementwisely: $J^*(v) = \sup_u \sum_i^n u_i v_i - u_i \text{log} u_i - \delta_{\triangle^{n-1}}(u)$. Let's consider the following minimization problem given v_i:

$$\begin{aligned} &\min_u \sum_i^n u_i \text{log} u_i - u_i v_i \\ &\text{s.t.} \mathbb{1} u = 1 \end{aligned}$$

where $\mathbb{1} = [1, \ldots, 1] \in \mathbb{R}^n$. It is obvious that this two problems share the same optimal variable u^* and the domain of log implies $u_i > 0$. Since the feasible set is compact and original energy function is continuous, the KKT condition holds on u^*. Therefore, we have certain $\lambda \in \mathbb{R}$ such that

$$\text{log} u_i^* + 1 - v_i + \lambda = 0, \ \forall i = 1, \ldots, n$$

which give $u_i^* = \exp\{-\lambda + v_i - 1\}$. Additionally, $\sum_{i=1}^n u_i^* = 1$. We can get

$$0 = \log(\sum_{i=1}^n \exp\{-\lambda + v_i - 1\}) = \log(\exp\{-\lambda - 1\} \sum_{i=1}^n e^{v_i}) = (-\lambda - 1) + \log(\sum_{i=1}^n e^{v_i})$$

Now, substitute u^* back into the convex conjugate and we can get

$$\begin{aligned} J(v)^* &= \sum_i^n \exp\{-\lambda + v_i - 1\}v_i - \exp\{-\lambda + v_i - 1\}(-\lambda + v_i - 1) \\ &= \sum_i^n -\exp\{-\lambda + v_i - 1\}(-\lambda - 1) \\ &= -(-\lambda - 1) = \log(\sum_{i=1}^n e^{v_i}) \end{aligned}$$

- Rewrite the convex conjugate as $J^*(v) = \sup_{\|u\|_2 \le \epsilon} \langle u, v\rangle - \frac{1}{2}\|u\|_2^2$. We first try to find the corresponding u^*.

$$\begin{aligned} u^* &= \text{argmin}_{\|u\|_2 \le \epsilon} \frac{1}{2}\|u\|_2^2 - \langle u, v\rangle + \frac{1}{2}\|v\|_2^2 \\ &= \text{argmin}_{\|u\|_2 \le \epsilon} \frac{1}{2}\|u - v\|_2^2 \end{aligned}$$

which is a projection problem i.e. project v into a convex set $\{u : \|u\|_2 \le \epsilon\}$. Therefore, if $\|v\|_2 \le \epsilon$, $u^* = v$. Otherwise, $u^* = \epsilon\frac{v}{\|v\|}$.

$$J^*(v) = \begin{cases} \frac{1}{2}\|v\|_2^2, & \|v\|_2 \le \epsilon \\ \epsilon\|v\|_2^2 - \frac{1}{2}\epsilon^2, & \text{otherwise} \end{cases}$$

Exercise 3 (4 points)**.** Assume $J : \mathbb{E} \to \mathbb{R}$, prove following facts of convex conjugate:

- $\tilde{J}(\cdot) = \alpha J(\cdot) \Rightarrow \tilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$, $\alpha > 0$.
- $\tilde{J}(\cdot) = J(\cdot - z) \Rightarrow \tilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z\rangle$.

Solution.

- Using the definition of convex conjugate:

$$\begin{aligned} \tilde{J}(\cdot) &= \sup_u \langle u, \cdot\rangle - \tilde{J}(u) \\ &= \sup_u \langle u, \cdot\rangle - \alpha J(u) \\ &= \alpha \underbrace{\sup_u \langle u, \cdot/\alpha\rangle - J(u)}_{J^*(\cdot/\alpha)} \\ &= \alpha J^*(\cdot/\alpha) \end{aligned}$$

- $\tilde{J}^*(\cdot) = \sup_u \langle u, \cdot\rangle - J(u - z)$. Define $v = u - z$ and by substitution we have:

$$\begin{aligned} \tilde{J}^*(\cdot) &= \sup_v \langle v + z, \cdot\rangle + J(v) \\ &= \sup_v \langle v, \cdot\rangle + J(v) + \langle z, \cdot\rangle \\ &= J^*(\cdot) + \langle \cdot, z\rangle \end{aligned}$$

Exercise 4 (6 points)**.** Show that projection onto a convex set is Lipschitz continuous with constant equals 1, i.e.

$$||\Pi_C(u) - \Pi_C(v)|| \leq ||u - v||, \ \forall u, v \in \mathbb{E}$$

where C is a convex set.

Solution. Given a point u, recall the property of projection:

$$\langle u - \Pi_C(u), x - \Pi_C(u) \rangle \leq 0, \ \forall x \in C.$$

Since $\Pi_C(v)$ is also an element in C, we get:

$$\langle u - \Pi_C(u), \Pi_C(v) - \Pi_C(u) \rangle \leq 0.$$

As same as above, we can get the ineuqlity for point v:

$$\langle v - \Pi_C(v), \Pi_C(u) - \Pi_C(v) \rangle \leq 0.$$

Sum above inequalities up, we have:

$$\begin{aligned}
&\langle u - \Pi_C(u) + \Pi_C(v) - v, \Pi_C(v) - \Pi_C(u) \rangle \leq 0 \\
\Rightarrow &\langle \Pi_C(v) - \Pi_C(u), \Pi_C(v) - \Pi_C(u) \rangle \leq \langle v - u, \Pi_C(v) - \Pi_C(u) \rangle \\
\Rightarrow &\, \|\Pi_C(v) - \Pi_C(u)\|^2 \leq \|v - u\| \, \|\Pi_C(v) - \Pi_C(u)\| \\
\Rightarrow &\, \|\Pi_C(v) - \Pi_C(u)\| \leq \|v - u\|
\end{aligned}$$