

Variational Methods for Computer Vision: Exercise Sheet 1

Exercise: October 23, 2019

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Refresher: Multivariate analysis.

- (a) For a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the *gradient* is defined as $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)^\top$. Calculate the gradients of the following functions.

i. $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = \frac{1}{2}\|x\|_2^2$,

ii. $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = \|x\|_2$.

Are there any points where the gradient is undefined?

- (b) For a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, the *Jacobian matrix* at the point $a \in \mathbb{R}^n$ is defined as

$$J_f(a) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Calculate the Jacobian matrix of the following functions:

i. $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2, f(r, \varphi) = (r \cos(\varphi), r \sin(\varphi))^\top$,

ii. $f: \mathbb{R} \rightarrow \mathbb{R}^2, f(t) = (r \cos(t), r \sin(t))^\top$.

- (c) For a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, the *divergence* is defined as $\operatorname{div} f = \sum_{i=1}^n \partial f_i/\partial x_i$. Calculate the divergence of the following functions:

i. $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x, y) = (-y, x)^\top$,

ii. $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x, y) = (x, y)^\top$.

- (d) For a function $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, the *curl* is defined as $\operatorname{curl} g = \partial g_2/\partial x - \partial g_1/\partial y$. Calculate the curl of function 1(c)i. Prove that the identity $\operatorname{curl}(\nabla f) = 0$ is true for any twice continuously differentiable function $f \in C^2(\mathbb{R}^2; \mathbb{R})$. Verify the identity with your result from 1(a)i.

- (e) When integrating a function $f: S \rightarrow \mathbb{R}$ over an open subset $S \subset \mathbb{R}^n$ using a parametrization $P \subset \mathbb{R}^n, \phi: P \rightarrow S$, the Jacobian of ϕ has to be taken into account as follows:

$$\int_S f(s) \, ds = \int_P f(\phi(p)) |\det J_\phi(p)| \, dp.$$

For $\gamma: [a, b] \rightarrow \mathbb{R}^n$, the line integral over a scalar field $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\|_2 \, dt$$

- i. Calculate the area enclosed by a circle of radius R .
ii. Calculate the circumference of a circle of radius R .

The results from task 1b might be helpful.

- (f) The divergence theorem (a special case of Stokes' theorem) states that an integral of the divergence of a function $f: S \rightarrow \mathbb{R}^n$ over a subset $S \subset \mathbb{R}^n$ can be replaced by an integral over the boundary ∂S of S :

$$\int_S \operatorname{div} f \, ds = \int_{\partial S} \langle f, n \rangle \, ds,$$

where $\langle \cdot, \cdot \rangle$ is the dot product and n the unit vector pointing in the direction normal to the boundary.

Convince yourself that this formula holds using f from task 1(c)ii and with S being a disk of radius R .

2. Convolutions and the Fourier transform.

- (a) Let $f, g, h \in L^1(\mathbb{R})$ be absolutely integrable functions. Consider the convolution of the functions f and g :

$$(f * g)(x) = \int_{\mathbb{R}} f(y) g(x - y) \, dy.$$

Show the following three algebraic identities:

- i. $(f * g) * h = f * (g * h)$
 - ii. $f * g = g * f$
 - iii. $f * (g + h) = f * g + f * h$
- (b) Let \mathcal{F} denote the Fourier transform operator:

$$\mathcal{F}\{f\} := \hat{f}(\nu) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \nu} \, dx.$$

Prove that the Fourier transform of the convolution of two functions is the same as the pointwise multiplication of the respective Fourier transforms:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}.$$

What implications does this have for computing the convolution?

- (c) Additionally let f and g be continuously differentiable. Show that:

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = \frac{dg}{dx} * f.$$

The results from 2a might be useful.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. Start MATLAB and visualize the vector fields from exercise 1(c)i and 1(c)ii. The commands `help meshgrid` and `help quiver` can be useful for that. Explain the intuition behind divergence-free and curl-free vector fields!
2. Download the archive `vmcv_ex01.zip` and unzip it on your home folder. In there should be a file named `coins.png`. Load the unzipped image using the following command:

```
f=double(imread('coins.png'));
```

Show the image using MATLAB's command:

```
figure; imshow(uint8(f));
```

3. Let W and H denote respectively the width and height of the input image f .
 - (a) Compute the gradient $\nabla f = (\partial_x^+ f, \partial_y^+ f)^\top$ of the image using the discretization scheme of forward differences:

$$(\partial_x^+ f)_{i,j} = \begin{cases} f_{i+1,j} - f_{i,j} & \text{if } i < W \\ 0 & i = W. \end{cases}$$
$$(\partial_y^+ f)_{i,j} = \begin{cases} f_{i,j+1} - f_{i,j} & \text{if } j < H \\ 0 & j = H. \end{cases}$$

Notice that the boundary values of the gradient are set to zero.

- (b) Compute the gradient without using any for loops this time. Can you tell the difference?
4. (a) Compute the convolution of the image with a Gaussian kernel. In theory, the Gaussian distribution is nonzero everywhere, however in practice we restrict ourself to truncated kernels. Set the radius of the kernel to $r = \text{ceil}(3 \times \sigma)$. The discrete convolution is given as:

$$g(i, j) = (w * f)(i, j) := \sum_{m=-r}^r \sum_{n=-r}^r w(m, n) f(i - m, j - n).$$

The discrete truncated Gaussian kernel can be written as follows:

$$w(m, n) \propto \exp\left(-\frac{m^2 + n^2}{2\sigma^2}\right)$$

In order to stay consistent with the continuous formulation of the Gaussian distribution make sure to normalize the kernel function such that the following holds:

$$\sum_{m=-r}^r \sum_{n=-r}^r w(m, n) = 1.$$

For simplicity you can ignore pixels where the mask goes beyond the edge of the image.

- (b) Find out how to obtain the same result as in (a) without using any loops. To this end, make yourself familiar with the `conv2` command.
 - (c) Now compute the gradient (see Ex. 3) of the image using convolution. Make sure to take care of the boundary values.
5. Let f_σ be the input image convolved with a Gaussian kernel of standard deviation σ . Compute the magnitude of the Gradient $|\nabla f_\sigma|$ for different values for σ . What do you observe?