Variational Methods for Computer Vision: Exercise Sheet 3

Exercise: November 13, 2019

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.
Reminder: A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex, if the following relation holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad \forall\, x, y \in \mathbb{R}^n,\ \lambda \in (0,1)$$

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. A point $\tilde{x} \in \mathbb{R}^n$ is a local minimizer of f if there exists a neighboorhood $\mathcal{N}(\tilde{x})$ such that $f(\tilde{x}) \leq f(x),\ \forall x \in \mathcal{N}(\tilde{x})$. A stationary point of f is a point at which the gradient vanishes, hence a point x^* which satisfies the following equation:

   $$\nabla f(x^*) = 0.$$

   Prove the following statements:

   (a) Every local minimizer of f is a global minimizer.

   (b) Suppose f is additionally differentiable. Every stationary point of f is a global minimizer.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. The epigraph of f is the following set:

   $$\text{epi } f := \{(u, a) \in \mathbb{R}^n \times \mathbb{R} \mid f(u) \leq a\}$$

   Prove that f is convex if and only if its epigraph is a convex set.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be real valued convex functions. Show whether or not the following functions are convex:

   (a)
   $$h(x) := \alpha f(x) + \beta g(x), \text{ where } \alpha, \beta > 0.$$

   (b)
   $$h(x) := \max(f(x), g(x))$$

   (c)
   $$h(x) := \min(f(x), g(x))$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable convex functions. Find a sufficient condition on f that assures the function h defined by

   $$h(x) := f(g(x))$$

   is convex by using the fact that function h is convex if and only if $h^{''}(x) \geq 0$.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the lecture we encountered the following cost function for denoising images:

$$E_\lambda(u) = \frac{1}{2}\sum_{i=1}^{N}(f_i - u_i)^2 + \frac{\lambda}{2}\sum_{i=1}^{N}\sum_{\substack{j\in\mathcal{N}(i)\\ j>i}}(u_i - u_j)^2. \tag{1}$$

   where u is the seeked image, f is the input image and where $\mathcal{N}(i)$ denotes a neighborhood of pixel i. Minimize the above function by solving the linear system of equations which arises from the optimality condition, using the Gauss-Seidel method.

2. To test the denoising capabilities of your method, degrade the input image with Gaussian noise (MATLAB: `help randn`). As a test image you can for example use the famous *camera man* test image, which also comes with MATLAB: `imread('cameraman.tif')`. Try out different initializations for the optimization. Does your result depend on the initialization? Explain why/why not. Also explain how the solution depends on the parameter λ.