Variational Methods for Computer Vision: Exercise Sheet 8

Exercise: December 18, 2019

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. **Piecewise Constant Mumford-Shah**
   Let $\Omega = [-5; 5] \times [-5; 5]$ be a rectangular area and let $I : \Omega \to [0, 1]$ be an image given by

   $$I(x, y) = \begin{cases} 1, & \text{if} \quad x^2 + y^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Furthermore let $C : [0, 1] \to \Omega$ be a curve represented by a circle centered at the origin having radius r.

   (a) Following the lecture, write down the gradient $\frac{\mathrm{d}E}{\mathrm{d}C}$ of the piecewise constant Mumford-Shah functional for two regions, external (ext) and internal (int),

   $$E(\{u_{\text{ext}}, u_{\text{int}}\}, C) = \int_{\Omega_{\text{ext}}} (I(x) - u_{\text{ext}})^2 \mathrm{d}x + \int_{\Omega_{\text{int}}} (I(x) - u_{\text{int}})^2 \mathrm{d}x + \nu |C|,$$

   for the two cases $r > 1$ and $r \leq 1$. The curvature κ of a circle with radius r is $\frac{1}{r}$ for all points on the circle.

   (b) Show that the gradient $\frac{\mathrm{d}E}{\mathrm{d}C}$ at $r = 1$ is not continuous. Why is $\nu \leq 1$ a good choice in order to obtain good segmentation results? What is the ideal choice for ν in our example?

2. **Piecewise Smooth Mumford-Shah**
   Derive the Euler-Lagrange equations (in u and w) of the Ambrosio-Tortorelli approximation of the piecewise smooth Mumford-Shah functional, i.e. compute both $\frac{\mathrm{d}E_\epsilon}{\mathrm{d}u}$ and $\frac{\mathrm{d}E_\epsilon}{\mathrm{d}w}$ for

   $$\begin{aligned} E_\epsilon(u, w) = &\int_\Omega (I(x) - u(x))^2 \, \mathrm{d}x + \lambda \int_\Omega w(x)^2 |\nabla u(x)|^2 \mathrm{d}x \\ &+ \nu \int_\Omega \left(\epsilon |\nabla w(x)|^2 + \frac{1}{4\epsilon} (w(x) - 1)^2 \right) \mathrm{d}x \end{aligned}$$

   Don't forget the boundary conditions.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. Implement the Ambrosio-Tortorelli approximation of piecewise smooth Mumford-Shah segmentation by alternatingly optimizing u and w. You can start from the provided script in `vmcv_ex08.zip`. The `.zip` folder also contains an example image for testing. Since the Euler-Lagrange equations are linear, you can solve the linear system exactly with Matlab's backslash operator. Alternatively, the exercise template suggests to use the Projected Conjugate Gradient method to solve the linear system for faster runtime (`help pcg`).