Variational Methods for Computer Vision: Solution Sheet 1

Exercise: 23 October 2019

Part I: Theory

1. **Refresher: Multivariate analysis.**

   (a) i. $\nabla f = (x, y)^\top$

   ii. $\nabla f = (x^2 + y^2)^{-1/2}(x, y)^\top$

   (b) i. $J = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}$

   ii. $J = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix}$

   (c) i. $\operatorname{div} f = 0$

   ii. $\operatorname{div} f = 2$

   (d) The solutions for the two functions from 1c are:

   i. $\operatorname{curl} f = 2$,

   ii. $\operatorname{curl} f = 0$.

   Proof for the curl of the gradient:

$$
\begin{aligned}
\operatorname{curl}(\nabla f) &= \operatorname{curl}\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y} && \text{(Symmetry of partial derivatives)} \\
&= 0.
\end{aligned}
$$

   (e) i. Using the coordinate transformation from 1(b)i with $\det J = r$, the area of a disk D_R of radius R can be calculated as

$$
\begin{aligned}
\iint_{D_R} \mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= 2\pi\left[\frac{1}{2}r^2\right]_0^R \\
&= \pi R^2.
\end{aligned}
$$

   ii. Using a parametrization like in 1(b)ii, $\gamma_R\colon [0, 2\pi] \to \mathbb{R}^2, \gamma_R(t) = (R\cos(t), R\sin(t))^\top$ with $\|\gamma_R'\|_2 = R$, the circumference of a circle with radius R can be calculated as

$$
\begin{aligned}
\int_{\gamma_R} \mathrm{d}s &= \int_0^{2\pi} R\,\mathrm{d}\varphi \\
&= 2\pi R.
\end{aligned}
$$

(f) First calculate the left-hand side of the divergence theorem:

$$\begin{aligned}\iint_{D_R} \operatorname{div} f \,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_R} 2\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\pi R^2. && \text{(Using 1(e)i)}\end{aligned}$$

For the right-hand side, first calculate the normal vector. The points on the boundary ∂D_R can be characterized by the zero set of $g(x, y) = x^2 + y^2 - R^2$. Calculating the gradient $\nabla g = (2x, 2y)^\top$ will give the direction of the normal n, and normalizing the gradient yields $n = (x^2 + y^2)^{-1/2}(x, y)^\top = (x, y)^\top/R$. Now the integral becomes

$$\begin{aligned}\int_{\partial D_R} \langle f, n\rangle \,\mathrm{d}s &= \int_{\gamma_R} \frac{1}{R}(x^2 + y^2)\,\mathrm{d}s \\ &= \int_{\gamma_R} R\,\mathrm{d}s \\ &= 2\pi R^2, && \text{(Using 1(e)ii)}\end{aligned}$$

which is equal to the left-hand side.

Remark: To compute normals on the boundary ∂P of a set $P \subset \mathbb{R}^n$, you can use the fact that the gradient is perpendicular to level sets. Define an implicit representation of P such that the boundary corresponds to the zero set, i.e. define a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $P = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \le 0\}$ and $\partial P = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0\}$. The normal at a point $\mathbf{x} \in \partial P$ corresponds to $\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$. It is pointing outwards, since f is negative inside and positive outside of P.

2. (a) i. Associativity:

$$\begin{aligned}((f * g) * h)(u) &= \int_{\mathbb{R}} (f * g)(x)\, h(u - x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)g(x - y)\,\mathrm{d}y\right) h(u - x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)g(x - y)h(u - x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)g(x - y)h(u - x)\,\mathrm{d}x\,\mathrm{d}y && \text{(Fubini's theorem)} \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x - y)h(u - x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g((x + y) - y)h(u - (x + y))\,\mathrm{d}x\,\mathrm{d}y && \text{(Translation invariance)} \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x)h(u - y - x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)(g * h)(u - y)\,\mathrm{d}y \\ &= (f * (g * h))(u).\end{aligned}$$

Remark: The translation invariance step can be seen as a special case of

$$\int_S f(s)\,\mathrm{d}s = \int_P f(\phi(p))|\det J_\phi(p)|\,\mathrm{d}p,$$

in the following way:

$$\begin{aligned}\int_{\mathbb{R}} g(x-y)\cdot h(u-x)\,\mathrm{d}x &= \int_{\mathbb{R}} (g\circ\varphi_1^y)(x)\cdot(h\circ\varphi_2^u)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (g\circ\varphi_1^y)(\varphi_y(x))\cdot(h\circ\varphi_2^u)(\varphi_y(x))\underbrace{|\det J_{\varphi_y}|}_{=1}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} g(x)h(u-y-x)\mathrm{d}x,\end{aligned}$$

with $\varphi_y(x) = x + y$, $\varphi_y(\mathbb{R}) = \mathbb{R}$, $\varphi_1^y(x) = x - y$, $\varphi_2^u(x) = u - x$.

ii. Commutativity:

$$\begin{aligned}(f*g)(u) &:= \int_{\mathbb{R}} f(x)\,g(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} g(\varphi_u(x))f(u-\varphi_u(x))|\det J_{\varphi_u}|\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(u-x)\,g(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} g(x)f(u-x)\,\mathrm{d}x \\ &=: (g*f)(u),\end{aligned}$$

with $\varphi_u(x) = u - x$, $|\det J_{\varphi_u}| = 1$, $\varphi_u(\mathbb{R}) = \mathbb{R}$.

iii. Distributivity:

$$\begin{aligned}f*(g+h)(u) &= \int_{\mathbb{R}} f(x)(g+h)(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)g(u-x) + f(x)h(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)g(u-x)\,\mathrm{d}x + \int_{\mathbb{R}} f(x)h(u-x)\,\mathrm{d}x \\ &= (f*g+f*h)(u).\end{aligned}$$

(b) We start with the definition of the Fourier transform:

$$\begin{aligned}\mathcal{F}\{f*g\}(\nu) &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right)e^{-2\pi i x\nu}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} g(x-y)e^{-2\pi i x\nu}\,\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

Introducing the substitution $z = x - y$, $\mathrm{d}z = \mathrm{d}x$ we arrive at

$$\begin{aligned}\int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} g(x-y)e^{-2\pi i x\nu}\,\mathrm{d}x\right)\mathrm{d}y &= \int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} g(z)e^{-2\pi i(z+y)\nu}\,\mathrm{d}z\right)\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)e^{-2\pi i y\nu}\int_{\mathbb{R}} g(z)e^{-2\pi i z\nu}\,\mathrm{d}z\,\mathrm{d}y \\ &= \underbrace{\int_{\mathbb{R}} f(y)e^{-2\pi i y\nu}\,\mathrm{d}y}_{=:\mathcal{F}\{f\}(\nu)}\;\underbrace{\int_{\mathbb{R}} g(z)e^{-2\pi i z\nu}\,\mathrm{d}z}_{=:\mathcal{F}\{g\}(\nu)}.\end{aligned}$$

As the Fourier transform and its inverse can be implemented to run in $\mathcal{O}(n \log n)$ time, convolutions of two images with n pixels each can be computed efficiently in $\mathcal{O}(n \log n)$ by exploiting this property:

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}.$$

The direct approach of implementing the convolution has runtime $\mathcal{O}(n^2)$.

(c) Let us consider the difference quotient

$$\frac{(f * g)(x+t) - (f * g)(x)}{t} = \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y.$$

Now taking the limit $t \to 0$ we have

$$\begin{aligned}
\frac{d}{dx}(f * g)(x) &= \lim_{t \to 0} \frac{(f * g)(x+t) - (f * g)(x)}{t} \\
&= \lim_{t \to 0} \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \lim_{t \to 0} f(y) \frac{g(x+t-y) - g(x-y)}{t} \, \mathrm{d}y \quad \text{(see \textbf{Remark 1})} \\
&= \int_{\mathbb{R}} f(y) (\frac{d}{dx} g)(x-y) \, \mathrm{d}y \quad \text{(see \textbf{Remark 2})} \\
&= f * \frac{dg}{dx} = \frac{dg}{dx} * f.
\end{aligned}$$

The remaining equality follows from the above and commutativity of convolution:

$$\frac{d}{dx}(f * g) = \frac{d}{dx}(g * f) = g * \frac{df}{dx} = \frac{df}{dx} * g.$$

Remark 1: In order to interchange integration and limit, one needs some additional conditions to hold (see Lebesgue's dominated convergence theorem). The theorem requires that

$$F_t(y) := f(y) \frac{g(x+t-y) - g(x-y)}{t},$$

convergences pointwise to a function $F_t(y) \to F(y)$ (1), and F_t is dominated by an integrable function $\bar{F}$ (2) in the sense

$$|F_t(y)| \leq \bar{F}(y), \forall t, \forall y.$$

(1) Pointwise convergence is easy to see, since g is continuously differentiable:

$$\lim_{t \to 0} F_t(y) = f(y) g'(x-y) \,.$$

(2) To find a dominating function, we can use the fact that g is integrable, i.e.

$$\lim_{|x| \to \infty} g(x) = 0 \quad \text{and thus} \quad \lim_{|x| \to \infty} g'(x) = 0 \,.$$

Since g' is continuous and tends to 0 for large $|x|$, there is an M such that $|g'(\xi)| \leq M$ for all $\xi \in \mathbb{R}$. From the mean value theorem, we further have

$$F_t(y) = f(y) g'(\xi) \quad \text{for some} \quad \xi \in [x-y, x-y+t] \,.$$

Thus, $|F_t(y)| \leq M|f(y)|$. Since f is integrable, $M|f|$ is also integrable, so it meets our requirements for the dominating function.

Remark 2: To see the equality

$$\lim_{t \to 0} \frac{g(x+t-y) - g(x-y)}{t} = (\frac{d}{dx} g)(x-y)$$

note that for any $z = \tilde{f}(x)$ we have

$$\lim_{t \to 0} \frac{g(\tilde{f}(x)+t) - g(\tilde{f}(x))}{t} = \lim_{t \to 0} \frac{g(z+t) - g(z)}{t} = g'(z) = g'(\tilde{f}(x)).$$