

Variational Methods for Computer Vision: Solution Sheet 6

Exercise: December 4, 2019

Part I: Theory

- Reminder: a linear map from X to Y is a function $L : X \rightarrow Y$ with the following properties

- (a) $L(u + v) = Lu + Lv, \forall u, v \in X;$
- (b) $L(\alpha u) = \alpha(Lu), \forall u \in X, \alpha \in \mathbb{R}.$

As the \tilde{e}_i form a basis, we can write Le_k uniquely as

$$Le_k = M_{1,k}\tilde{e}_1 + \dots + M_{m,k}\tilde{e}_m,$$

for all $k \in \{1, \dots, n\}$. These scalars $M_{i,j}$ then completely determine the linear map L . The matrix

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,n} \\ \vdots & \ddots & \vdots \\ M_{m,1} & \cdots & M_{m,n} \end{pmatrix} = \begin{pmatrix} | & & | \\ [L(e_1)]_{\tilde{e}} & \cdots & [L(e_n)]_{\tilde{e}} \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}$$

is then the so called *matrix representation* of L with respect to the bases $\{e_1, \dots, e_n\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_m\}$.

We verify that for some $x = \alpha_1 e_1 + \dots + \alpha_n e_n$ we have

$$[L(x)]_{\tilde{e}} = \sum_{j=1}^n \alpha_j [L(e_j)]_{\tilde{e}} = \sum_{j=1}^n \alpha_j M_{.,j} = Mx.$$

- We start by the directional derivative, as on the last sheets:

$$\begin{aligned} \frac{dE(u)}{du} \Big|_h &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (E(u + \varepsilon h) - E(u)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} (\mathcal{L}(u + \varepsilon h, \nabla(u + \varepsilon h), A(u + \varepsilon h)) - \mathcal{L}(u, \nabla u, Au)) \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} (\mathcal{L}(u + \varepsilon h, \nabla u + \varepsilon \nabla h, Au + \varepsilon Ah) - \mathcal{L}(u, \nabla u, Au)) \, dx \\ &= \int_{\Omega} \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial \nabla u} \nabla h + \frac{\partial \mathcal{L}}{\partial Au} (Ah) \right) \, dx \\ &= \int_{\Omega} \left(\frac{\partial \mathcal{L}}{\partial u} h - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} h + A^* \frac{\partial \mathcal{L}}{\partial Au} h \right) \, dx + \int_{\partial \Omega} h \left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle \, ds \end{aligned}$$

Hence the Euler-Lagrange equation is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} + A^* \frac{\partial \mathcal{L}}{\partial Au} &= 0, \text{ for } x \in \Omega \\ \left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle &= 0, \text{ for } x \in \partial \Omega, n \text{ is the normal.} \end{aligned}$$