Variational Methods for Computer Vision: Solution Sheet 8

Exercise: December 18, 2019

Part I: Theory

1. Recall from the lecture, that the Euler-Lagrange equation for the two-region Mumford-Shah functional for a curve $C : [0, 1] \to \Omega \subset \mathbb{R}^2$ and image $I : \Omega \to \mathbb{R}$ is given by

$$\frac{\mathrm{d}E}{\mathrm{d}C} = \left((I - u_{\text{int}})^2 - (I - u_{\text{ext}})^2 + \nu\kappa\right) n_C. \tag{1}$$

Here, u_{int} and u_{ext} are the average intensities inside and outside the curve C, i.e.,

$$u_{\text{int}} = \frac{\int_{\text{int}(C)} I(x)\mathrm{d}x}{\int_{\text{int}(C)} \mathrm{d}x},\ u_{\text{ext}} = \frac{\int_{\text{ext}(C)} I(x)\mathrm{d}x}{\int_{\text{ext}(C)} \mathrm{d}x}. \tag{2}$$

We will consider the curve evolution

$$\frac{\partial C}{\partial t} = -\frac{\mathrm{d}E}{\mathrm{d}C} = \left(-(I - u_{\text{int}})^2 + (I - u_{\text{ext}})^2 - \nu\kappa\right) n_C. \tag{3}$$

Intuitively, we evolve the curve along the normal vector n_C depending on the sign of the term in the brackets.

(a) The curvature κ of a circle with radius r is $\kappa = \frac{1}{r}$. We can use this fact in calculating the Euler-Lagrange equations for the 2 different cases.

Case $r > 1$:

$$u_{\text{ext}} = 0, \quad u_{\text{int}} = \frac{\pi}{\pi r^2} = \frac{1}{r^2}.$$

This leads to following inner term:

$$(I - u_{\text{ext}})^2 - (I - u_{\text{int}})^2 - \nu\kappa = (0 - 0)^2 - \left(0 - \frac{1}{r^2}\right)^2 - \frac{\nu}{r} = -\frac{1}{r^4} - \frac{\nu}{r}.$$

Case $r \leq 1$:

$$u_{\text{ext}} = \frac{\pi - \pi r^2}{100 - \pi r^2}, \quad u_{\text{int}} = 1.$$

A short computation shows:

$$\begin{aligned}(I - u_{\text{ext}})^2 - (I - u_{\text{int}})^2 - \nu\kappa &= \left(1 - \frac{\pi - \pi r^2}{100 - \pi r^2}\right)^2 - 0 - \frac{\nu}{r} \\ &= \left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r}.\end{aligned}$$

(b) We see that the limits differ,

$$\lim_{r \searrow 1} -\frac{1}{r^4} - \frac{\nu}{r} = -1 - \nu,$$
$$\lim_{r \nearrow 1} \left(\frac{100 - \pi}{100 - \pi r^2} \right)^2 - \frac{\nu}{r} = \left(\frac{100 - \pi}{100 - \pi} \right)^2 - \nu = 1 - \nu,$$

hence, the functional derivative at $r = 1$ is not continuous.

This shows that the original energy $E(C)$ is not differentiable, which can lead to convergence problems when using gradient descent-type algorithms as they technically require differentiability of the energy.

$\nu \leq 1$ is a good choice because it ensures that the curve evolves in the right direction for both cases $r > 1$ and $r \leq 1$. $\nu = 0$ can be considered optimal in some sense, because that minimizes the difference in the magnitude of the gradient direction for the two cases $r > 1$ and $r \leq 1$ (they are both 1).

2. Let us first consider the energy as a functional of u:

$$E(u) = \int_\Omega \mathcal{L}(u, \nabla u) \mathrm{d}x.$$

Using the result from the lecture for energies of this form, the optimality condition is

$$\frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} = 2(u - I) - 2\lambda \operatorname{div}(w^2 \nabla u) = 0 \qquad \text{in } \Omega,$$
$$\left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle = \left\langle 2\lambda w^2 \nabla u, n \right\rangle = 2\lambda w^2 \langle \nabla u, n \rangle = 0 \qquad \text{on } \partial\Omega.$$

As usual n is the normal on the boundary. Notice that the energy is quadratic in both u and ∇u, which means that we have a linear Euler-Lagrange equation. For image denoising von Neumann boundary conditions are appropriate, i.e. $\langle \nabla u, n \rangle = 0$. This implies that the boundary term of the Euler-Lagrange equation is fulfilled.

Next, we consider the energy as a functional of w:

$$E(w) = \int_\Omega \mathcal{L}(w, \nabla w) \mathrm{d}x,$$

and compute the optimiality condition analogously as

$$\frac{\partial \mathcal{L}}{\partial w} - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla w} = 2\lambda |\nabla u|^2 w + \frac{\nu}{2\epsilon}(w - 1) - 2\nu\epsilon \Delta w = 0 \qquad \text{in } \Omega,$$
$$\left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle = \langle 2\nu\epsilon \nabla w, n \rangle = 2\nu\epsilon \langle \nabla w, n \rangle = 0 \qquad \text{on } \partial\Omega.$$

Again, this equation is linear in w. The boundary term is fulfilled with von Neumann boundary conditions ($\langle \nabla w, n \rangle = 0$) and this choice also seems appropriate for the application (imagine the border indicater function w to extend with constant value at the border in normal direction).

In the implementation we alternatingly optimize u and w. In each step we can directly compute the optimal u and w by solving a linear equation system.