Variational Methods for Computer Vision: Solution Sheet 9

Exercise: January 8, 2020

Part I: Theory

1. Using the property that the delta distribution $\delta(\phi)$ is the derivative of H, we get

$$|\nabla H(\phi)| = |\delta(\phi) \cdot \nabla \phi(x)| = \delta(\phi)|\nabla \phi(x)|.$$

So we can rewrite the Lagrangian in the following form:

$$\begin{aligned}\mathcal{L} &= f_1 \cdot H(\phi) + f_2(1 - H(\phi)) + \nu|\nabla H(\phi)| \\ &= (f_1 - f_2)H(\phi) + \nu\delta(\phi)|\nabla\phi(x)| + f_2.\end{aligned}$$

So

$$\frac{\partial \mathcal{L}}{\partial \nabla \phi} = \frac{\partial}{\partial \nabla \phi}\nu|\nabla\phi|\delta(\phi) = \nu \cdot \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = H'(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)| = \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)|.$$

Using both results we can compute

$$\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla \phi}\right) = \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)| - \nu \operatorname{div}\left(\delta(\phi) \cdot \frac{\nabla\phi}{|\nabla\phi|}\right).$$

For further simplification we use the property

$$\operatorname{div}(fV) = \langle \nabla f, V\rangle + f \operatorname{div}(V)$$

for a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ and a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ with $f = \delta(\phi)$ and $V = \frac{\nabla\phi}{|\nabla\phi|}$:

$$\Rightarrow \quad \frac{\partial E}{\partial \phi} = \delta(\phi)(f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi| - \nu\left\langle \nabla(\delta(\phi)), \frac{\nabla\phi}{|\nabla\phi|}\right\rangle - \delta(\phi)\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right).$$

The inner product can be rewritten using the chain rule and $\langle\nabla\phi, \nabla\phi\rangle = |\nabla\phi|^2$:

$$\left\langle \nabla(\delta(\phi)), \frac{\nabla\phi}{|\nabla\phi|}\right\rangle = \delta'(\phi)\left\langle \nabla\phi, \frac{\nabla\phi}{|\nabla\phi|}\right\rangle = \delta'(\phi)\frac{|\nabla\phi|^2}{|\nabla\phi|} = \delta'(\phi)|\nabla\phi|.$$

Thus, the terms $\nu\delta'(\phi)|\nabla\phi|$ cancel and we finally obtain

$$\frac{\partial E}{\partial \phi} = \delta(\phi)(f_1 - f_2) - \nu\delta(\phi)\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right).$$

2.

$$E(\phi) = \int g(x)|\nabla H(\phi(x))| \mathrm{d}x = \int g(x)\delta(\phi)|\nabla\phi| \mathrm{d}x.$$

With $\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla\phi}\right)$ this leads to

$$\frac{\partial \mathcal{L}}{\partial \phi} = \delta'(\phi) g(x) |\nabla\phi|$$

and

$$\frac{\partial \mathcal{L}}{\partial \nabla\phi} = g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla(\phi)|}.$$

Combining both results and further evaluating $\operatorname{div}\left(g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla(\phi)|}\right)$ gives

$$\begin{aligned}
\frac{\partial E}{\partial \phi} &= \frac{\partial L}{\partial \phi} - \operatorname{div}\left(\frac{\partial L}{\partial \nabla\phi}\right) \\
&= \delta'(\phi) g |\nabla\phi| - \operatorname{div}\left(g\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right) \\
&= \delta'(\phi) g |\nabla\phi| - \delta'(\phi) g \frac{\langle\nabla\phi, \nabla\phi\rangle}{|\nabla\phi|} - \delta(\phi) \operatorname{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right) \\
&= -\delta(\phi) \operatorname{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right).
\end{aligned}$$