

# Variational Methods for Computer Vision: Solution Sheet 10

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Exercise: January 15, 2020

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## Part I: Theory

1. (a) We use the fact that for two vectors  $a$  and  $b$  in  $\mathbb{R}^n$ ,

$$|a + b| \leq |a| + |b|,$$

and rewrite the energy as

$$E(u) = \int_{\Omega} (f_1 - f_2)u + \nu|\nabla u| + f_2 \, dx.$$

Then a short calculation shows for functions  $u, v$  and  $\alpha \in [0, 1]$

$$\begin{aligned} E(\alpha u + (1 - \alpha)v) &= \\ \int_{\Omega} (f_1 - f_2)(\alpha u + (1 - \alpha)v) + \nu|\alpha \nabla u + (1 - \alpha)\nabla v| + \alpha f_2 + (1 - \alpha)f_2 \, dx &\leq \\ \int_{\Omega} \alpha(f_1 - f_2)u + (1 - \alpha)(f_1 - f_2)v + \alpha\nu|\nabla u| + (1 - \alpha)\nu|\nabla v| + \alpha f_2 + (1 - \alpha)f_2 \, dx &= \\ \alpha E(u) + (1 - \alpha)E(v). \end{aligned}$$

Thus,  $E$  is convex in  $u$ .

- (b)  $[0, 1] \subset \mathbb{R}$  is a convex set (which you can show directly by considering a convex combination of two elements of  $[0, 1]$ ). Therefore it holds for  $u, v \in U$ ,  $x \in \Omega$  and  $\alpha \in [0, 1]$

$$\alpha u(x) + (1 - \alpha)v(x) \in [0, 1].$$

Since  $x \in \Omega$  can be chosen arbitrary, it follows that  $\alpha u + (1 - \alpha)v \in U$ . Therefore  $U$  is a convex set.

Note: In general, it holds that for any domain  $\Omega$ ,

$$U := \{u : \Omega \rightarrow C\} \text{ convex} \Leftrightarrow C \text{ convex}.$$

- (c) Let  $F(u) = \int_{\Omega} (f(x) - u(x))^2 dx$ . Then for all  $u \in U$

$$\begin{aligned} F(u) &= \int_{f(x)>1} (f(x) - u(x))^2 dx + \int_{f(x)<0} (f(x) - u(x))^2 dx + \int_{f(x) \in [0,1]} (f(x) - u(x))^2 dx \\ &\geq \int_{f(x)>1} (f(x) - 1)^2 dx + \int_{f(x)<0} f(x)^2 dx + 0 = F(f_U), \end{aligned}$$

which implies  $f_U$  is the global minimum of  $F(u)$  and therefore the projection of  $f$  onto the convex set  $U$ .

(d) Using the result from the previous exercise sheets

$$\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} = 0.$$

The partial derivatives are given by:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} &= f_1 - f_2, \\ \frac{\partial \mathcal{L}}{\partial \nabla u} &= \nu \frac{\nabla u}{|\nabla u|}. \end{aligned}$$