Chapter 1
Convex Analysis

Convex Optimization for Machine Learning & Computer Vision
WS 2019/20

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Last updated: 21.10.2019

Convex Set

Convex Optimization

Notations

- $\mathbb{E}$ is a *Euclidean space* (i.e., finite dimensional inner product space), equipped with
  - ① Inner product $\langle \cdot, \cdot \rangle$, e.g., $\langle u, v \rangle = u^\top v$ if $\mathbb{E} = \mathbb{R}^n$;
  - ② Norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ satisfying polarization identity:

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2.$$

- C is a closed, convex subset of $\mathbb{E}$.
- J is a convex *objective* function.

Convex Optimization

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Convex optimization

$$\text{minimize } J(u) \quad \text{over } u \in C.$$

First questions:

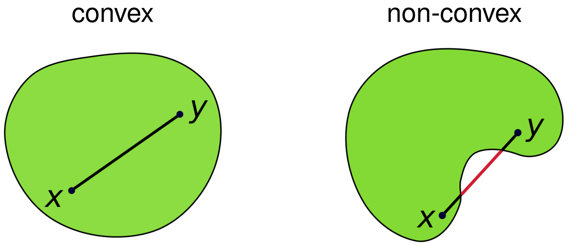
- What is a convex set?
- What is a convex function?

Convex set

Definition

A set C is said to be **convex** if

$$\alpha u + (1-\alpha) v \in C, \quad \forall u, v \in C,\ \forall \alpha \in [0,1].$$


convex

non-convex

Recall basic concepts in analysis

Definition

- A set $C \subset \mathbb{E}$ is **open** if $\forall u \in C, \exists \epsilon > 0$ s.t. $B_\epsilon(u) \subset C$, where $B_\epsilon(u) := \{v \in \mathbb{E} : \|v - u\| < \epsilon\}$.
- A set $C \subset \mathbb{E}$ is **closed** if its complement $\mathbb{E} \backslash C$ is open.
- The **closure** of a set $C \subset \mathbb{E}$ is

$$\operatorname{cl} C = \{u \in \mathbb{E} : \exists \{u^k\} \subset C \text{ s.t. } \lim_{k \to \infty} u^k = u\}.$$

- The **interior** of a set $C \subset \mathbb{E}$ is

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- The **interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{int} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

- The **relative interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{rint} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \cap \operatorname{aff} C \subset C\},$$

  with aff C the **affine hull** of C. If C is a *convex* set, then

$$\operatorname{rint} C = \{u \in C : \forall v \in C, \exists \alpha > 1 \text{ s.t. } v + \alpha(u - v) \in C\}.$$

Basic properties

The following operations preserve the convexity:

- Intersection: $C_1 \cap C_2$.
- Summation: $C_1 + C_2 := \{u^1 + u^2 : u^1 \in C_1, u^2 \in C_2\}$.
- Closure: $\operatorname{cl} C$.
- Interior and relative interior: $\operatorname{int} C$, $\operatorname{rint} C$.

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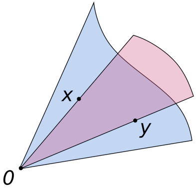
Convex cone

C is a **cone** if $C = \alpha C$ for any $\alpha > 0$.
C is a **convex cone** if C is a cone and is convex as well.



Convex cone.

Separation of convex sets

Theorem (separation of convex sets)

Let C_1, C_2 be nonempty convex subsets of $\mathbb{E}$.

1. Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E} \backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$
2. Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E} \backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$
3. Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$
4. Assume $\emptyset \neq \operatorname{int} C_1 \subset \mathbb{E} \backslash C_2$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$