Chapter 1
Convex Analysis

Convex Optimization for Machine Learning & Computer Vision
WS 2019/20

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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Convex Set

Convex Set
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Convex Conjugate
Duality Theory
Proximal Operator

Convex Optimization

Notations

- $\mathbb{E}$ is a *Euclidean space* (i.e., finite dimensional inner product space), equipped with
  1. Inner product $\langle \cdot, \cdot \rangle$, e.g., $\langle u, v \rangle = u^\top v$ if $\mathbb{E} = \mathbb{R}^n$;
  2. Norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ satisfying polarization identity:

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2.$$

- C is a closed, convex subset of $\mathbb{E}$.
- J is a convex *objective* function.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Convex optimization

$$\text{minimize } J(u) \quad \text{over } u \in C.$$

First questions:

- What is a convex set?
- What is a convex function?

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

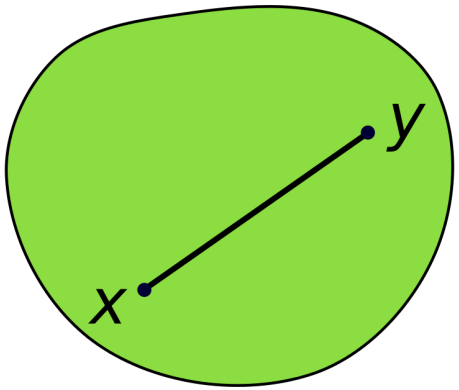
Convex set

Definition

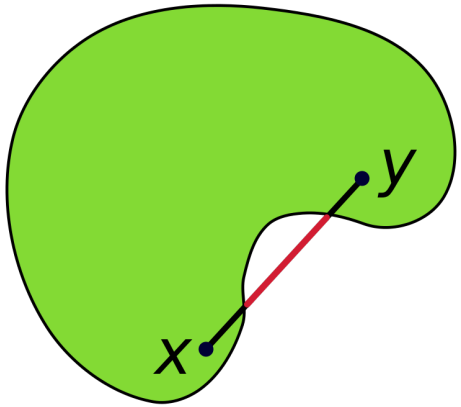
A set C is said to be **convex** if

$$\alpha u + (1-\alpha)v \in C, \quad \forall u, v \in C, \ \forall \alpha \in [0,1].$$

convex



non-convex



Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Recall basic concepts in analysis

Definition

- A set $C \subset \mathbb{E}$ is **open** if $\forall u \in C$, $\exists \epsilon > 0$ s.t. $B_\epsilon(u) \subset C$, where $B_\epsilon(u) := \{v \in \mathbb{E} : \|v - u\| < \epsilon\}$.
- A set $C \subset \mathbb{E}$ is **closed** if its complement $\mathbb{E} \backslash C$ is open.
- The **closure** of a set $C \subset \mathbb{E}$ is

$$\text{cl}\, C = \{u \in \mathbb{E} : \exists \{u^k\} \subset C \text{ s.t. } \lim_{k \to \infty} u^k = u\}.$$

- The **interior** of a set $C \subset \mathbb{E}$ is

$$\text{int}\, C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- The **interior** of a set $C \subset \mathbb{E}$ is

$$\text{int}\, C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

- The **relative interior** of a set $C \subset \mathbb{E}$ is

$$\text{rint}\, C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \cap \text{aff}\, C \subset C\},$$

with aff C the **affine hull** of C. If C is a *convex* set, then

$$\text{rint}\, C = \{u \in C : \forall v \in C, \exists \alpha > 1 \text{ s.t. } v + \alpha(u - v) \in C\}.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Basic properties

The following operations preserve the convexity:

- Intersection: $C_1 \cap C_2$.
- Summation: $C_1 + C_2 := \{u^1 + u^2 : u^1 \in C_1, u^2 \in C_2\}$.
- Closure: cl C.
- Interior and relative interior: int C, rint C.

In general, the union of convex sets is not convex.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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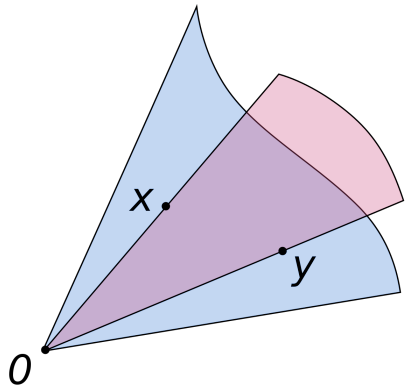
Convex cone

C is a **cone** if $C = \alpha C$ for any $\alpha > 0$.
C is a **convex cone** if C is a cone and is convex as well.



Convex cone.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Separation of convex sets

Theorem (separation of convex sets)

Let C_1, C_2 be nonempty convex subsets of $\mathbb{E}$.

1. Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E} \backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$
2. Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E} \backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$
3. Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$
4. Assume $\emptyset \neq \operatorname{int} C_1 \subset \mathbb{E} \backslash C_2$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Convex Function

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Convex functions

- An **extended real-valued function** J maps from $\mathbb{E}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.
- The **domain** of $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{dom} J = \{u \in \mathbb{E} : J(u) < \infty\}.$$

- The function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is **proper** if $\operatorname{dom} J \neq \emptyset$.

Definition

We say $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a **convex function** if

1. $\operatorname{dom} J$ is a convex set.
2. For all $u, v \in \operatorname{dom} J$ and $\alpha \in [0, 1]$ it holds that

$$J(\alpha u + (1 - \alpha)v) \leq \alpha J(u) + (1 - \alpha)J(v).$$

We say J is **strictly convex** if the above inequality is strict for all $\alpha \in (0, 1)$ and $u \neq v$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Examples

- $J_{data}(u) = \|u - z\|_p^p$, where $p \geq 1$ and $\|\cdot\|_p$ is ℓ^p-norm.
- $J_{regu}(u) = \|Ku\|_q^q$, where K is linear transform and $q \geq 1$.
- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$, where $\alpha > 0$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Negative binary entropy ($\epsilon > 0$):
  $J_\epsilon(u) = \epsilon\big(u \log(u) + (1-u)\log(1-u)\big)$.
- Soft plus: $J_\epsilon(v) = \epsilon \log(1 + \exp(v/\epsilon))$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Soft plus: $J_\epsilon(v) = \epsilon \log(1 + \exp(v/\epsilon))$.
- **Indicator function** of a convex set $C \subset \mathbb{E}$:

$$\delta_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{otherwise.} \end{cases}$$

Formulate *constrained optimization* with indicator function:

$$\min J(u) \text{ over } u \in C. \quad \leftrightarrow \quad \min J(u) + \delta_C(u) \text{ over } u \in \mathbb{E}.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Basic facts

(As exercises)

- Any norm (over a normed vector space) is a convex function.
- J is a convex function and A is an affine transform $\Rightarrow u \mapsto J(A(u))$ is a convex function.
- (Jensen's inequality) $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex iff

$$J\left(\sum_{i=1}^{n} \alpha_i u^i\right) \leq \sum_{i=1}^{n} \alpha_i J(u^i),$$

whenever $\{u^i\}_{i=1}^n \subset \mathbb{E}$, $\{\alpha_i\}_{i=1}^n \subset [0,1]$, $\sum_{i=1}^n \alpha_i = 1$.

(Hence it is an equivalent definition of a convex function.)

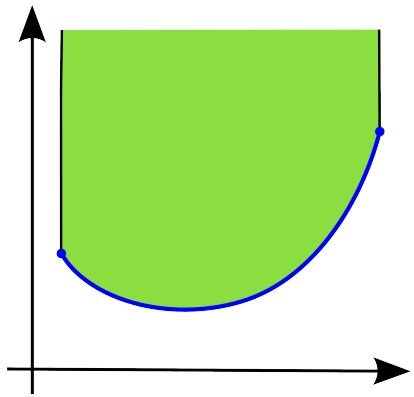
Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Epigraph

Definition

The **epigraph** of a proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{epi} J = \{(u, \alpha) \in \mathbb{E} \times \mathbb{R} : J(u) \leq \alpha\}.$$



epigraph

Theorem

A proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex (resp. strictly convex) iff $\operatorname{epi} J$ is a convex (resp. strictly convex) set.

Proof: as exercise.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Lipschitz continuity

Definition

Assume $J : \mathbb{E} \to \overline{\mathbb{R}}$ with $\operatorname{rint}\operatorname{dom} J \neq \emptyset$. We say J is **locally Lipschitz** at $u \in \operatorname{rint}\operatorname{dom} J$ with modulus $L_u > 0$ if there exists $\epsilon > 0$ s.t.

$$|J(u^1) - J(u^2)| \leq L_u \|u^1 - u^2\| \quad \forall u^1, u^2 \in B_\epsilon(u) \cap \operatorname{rint}\operatorname{dom} J.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Theorem

A proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \operatorname{rint}\operatorname{dom} J$.

Proof: found in script.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Existence of Minimizer

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Global vs. Local minimizer

Recall the optimization of $J : \mathbb{E} \to \overline{\mathbb{R}}$:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Definition

1. $u^* \in \mathbb{E}$ is a **global minimizer** if $J(u^*) \leq J(u)$ for all $u \in \mathbb{E}$.
2. u^* is a **local minimizer** if $\exists \epsilon > 0$ s.t. $J(u^*) \leq J(u)$ for all $u \in B_\epsilon(u^*)$.
3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by $J(u^*) < J(u)$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by $J(u^*) < J(u)$.

Theorem

For any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, if $u^* \in \operatorname{dom} J$ is a local minimizer of J, then it is also a global minimizer.

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Does a minimizer always exist?

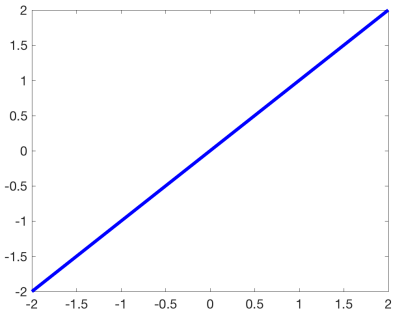
- Consider

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E},$$

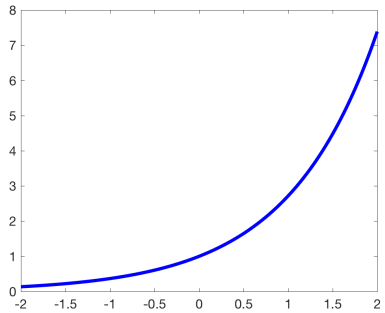
where $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a proper, convex function.

- Some counterexamples for $J : \mathbb{R} \to \overline{\mathbb{R}}$:

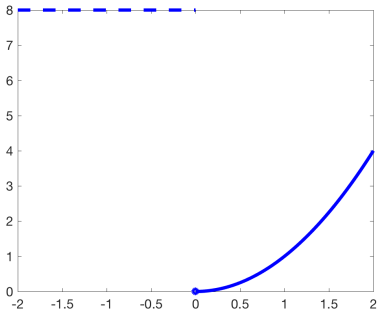
u



$\exp u$



$u^2 + \delta\{u > 0\}$



- Next we formalize our observations and derive sufficient conditions for existence.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

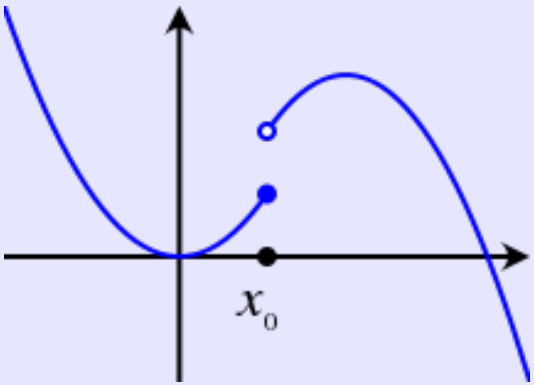
Sufficient conditions for existence

Definition

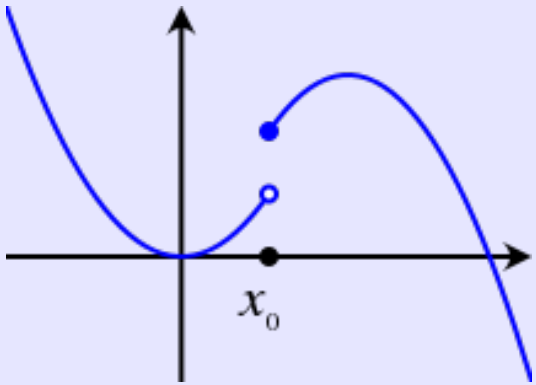
1. J is **bounded from below** if $J(\cdot) \geq C$ for some $C \in \mathbb{R}$.
2. J is **coercive** if $J(u) \to \infty$ whenever $\|u\| \to \infty$.
   - Proposition: J is coercive if $\operatorname{dom} J$ is bounded.
3. J is **lower semi-continuous** (lsc) at u^* if

$$J(u^*) \leq \liminf_{k \to \infty} J(u^k), \text{ whenever } u^k \to u^*.$$

lsc



not lsc



- Proposition: J is lsc iff $\operatorname{epi} J$ is closed.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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$$J(u^*) \leq \liminf_{k \to \infty} J(u^k), \text{ whenever } u^k \to u^*.$$

Theorem

Any proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc (everywhere), has a (global) minimizer.

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Uniqueness

- Recall that a function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is strictly convex if

$$J(\alpha u + (1-\alpha)v) < \alpha J(u) + (1-\alpha)J(v),$$

for all $u, v \in \operatorname{dom} J,\ u \neq v,\ \alpha \in (0,1)$.

Theorem

The minimizer of a strictly convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is unique.

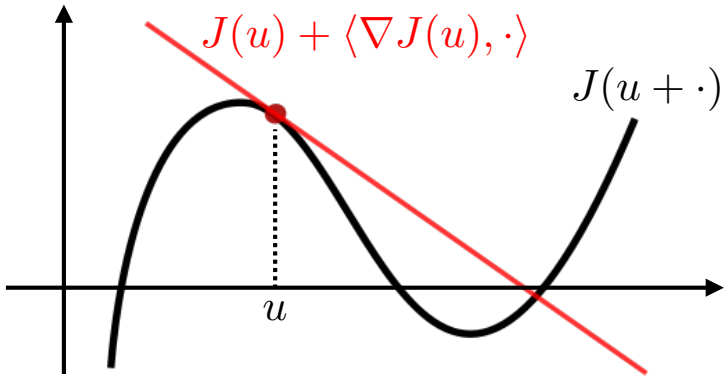
Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Subdifferential

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Differential



Definition

$J : \mathbb{E} \to \overline{\mathbb{R}}$ is called (Fréchet) **differentiable** at $u \in \operatorname{int} \operatorname{dom} J$ and $\nabla J(u) \in \mathbb{E}$ is the (Fréchet) **differential** of J at u if

$$\lim_{h \to 0} \frac{|J(u+h) - J(u) - \langle \nabla J(u), h \rangle|}{\|h\|} = 0.$$

J is **continuously differentiable** at $u \in \operatorname{int} \operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $(\operatorname{int} \operatorname{dom} J) \cap B_\epsilon(u)$ for some $\epsilon > 0$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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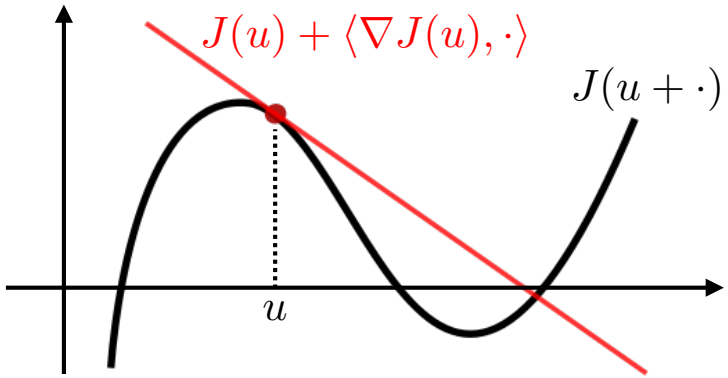

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Remark

For a convex function J, differentiability of J can be defined by considering $J : \operatorname{aff} \operatorname{dom} J \to \overline{\mathbb{R}}$ and $u \in \operatorname{rint} \operatorname{dom} J$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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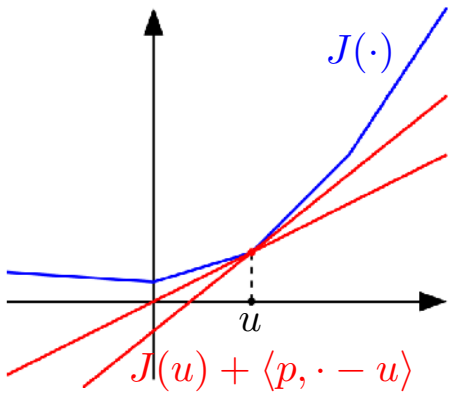
The **subdifferential** of a convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ at $u \in \operatorname{dom} J$ is defined by

$$\partial J(u) = \{p \in \mathbb{E} : J(v) \geq J(u) + \langle p, v - u \rangle \ \forall v \in \mathbb{E}\}.$$



Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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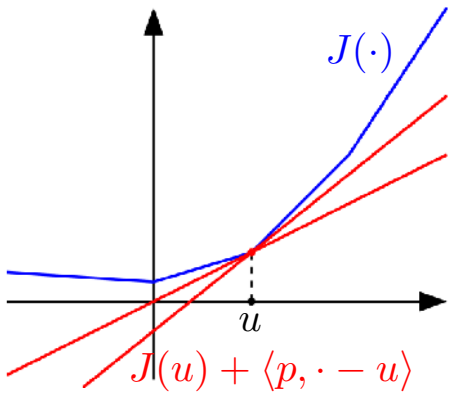
$$\partial J(u) = \{p \in \mathbb{E} : J(v) \geq J(u) + \langle p, v - u \rangle \ \forall v \in \mathbb{E}\}.$$



Geometric interpretation

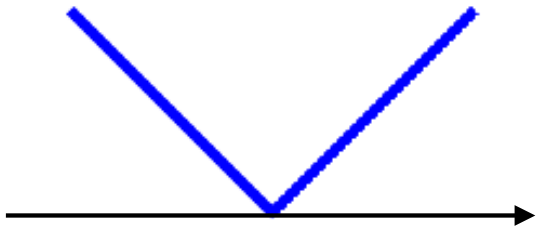
$p \in \partial J(u)$ iff $(p, -1)$ is a normal vector for the supporting hyperplane of epi J at $(u, J(u))$, i.e.,

$$\left\langle \begin{bmatrix} p \\ -1 \end{bmatrix}, \begin{bmatrix} u \\ J(u) \end{bmatrix} \right\rangle \geq \left\langle \begin{bmatrix} p \\ -1 \end{bmatrix}, \begin{bmatrix} v \\ \alpha \end{bmatrix} \right\rangle, \quad \forall (v, \alpha) \in \operatorname{epi} J.$$

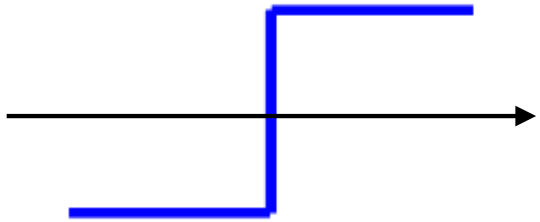
Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Subdifferential: Examples

1. $J(u) = |u|$.



$J(u)$



$\partial J(u)$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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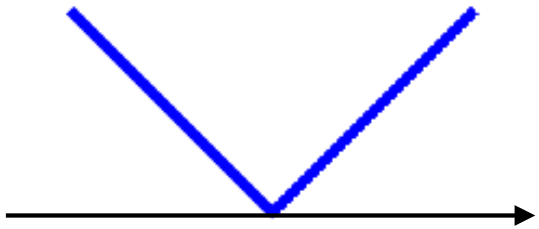

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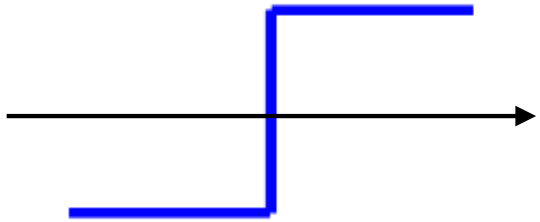

$\partial J(u)$

2. Given a closed, convex subset $C \subset \mathbb{E}$ and $u \in C$,

$$\partial \delta_C(u) = \{p \in \mathbb{E} : \langle p, v - u \rangle \leq 0 \ \forall v \in C\} =: N_C(u),$$

known as the *normal cone* of C at u.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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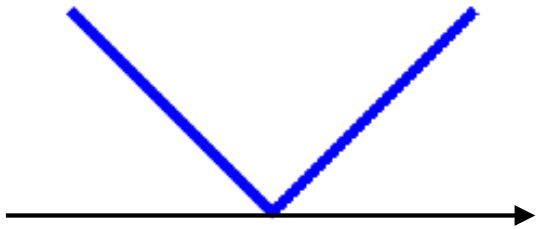

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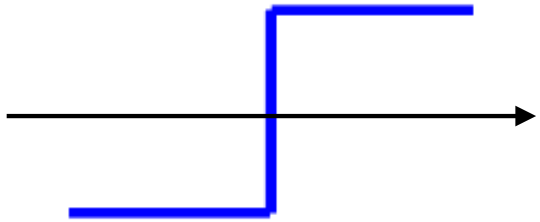

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known as the *normal cone* of C at u.

3. $J(u) = |||u||| \ \Rightarrow \ \partial J(0) = \{p : |||p|||_* \leq 1\}$. $|||\cdot|||_*$ is the *dual norm* of $|||\cdot|||$, i.e., $|||p|||_* = \sup\{\langle p, u \rangle : |||u||| \leq 1\}$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Subdifferential calculus

Theorem (chain rule under linear transform)

Let $\widetilde{J}(\cdot) = J(K\cdot)$ with some convex function J and linear transform K. Then

$$\partial\widetilde{J}(u) = K^\top \partial J(Ku)$$

whenever $Ku \in \operatorname{rint}\operatorname{dom} J$.

Example: $J(u) = \|Ku\| \Rightarrow \partial J(u) = K^\top \partial\|\cdot\|(Ku)$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Theorem (summation rule)

Let $\widetilde{J}(\cdot) = J_1(\cdot) + J_2(\cdot)$, where J_1, J_2 are convex functions s.t.

$$\operatorname{rint}\operatorname{dom} J_1 \cap \operatorname{rint}\operatorname{dom} J_2 \neq \emptyset.$$

Then for any $u \in \operatorname{dom} J_1 \cap \operatorname{dom} J_2$, we have

$$\partial\widetilde{J}(u) = \partial J_1(u) + \partial J_2(u).$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Properties of subdifferential map

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \operatorname{int} \operatorname{dom} J$, $\partial J(u)$ is a nonempty, compact, and convex subset.

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then ∂J is a **monotone operator**, i.e. $\forall u^1, u^2 \in \operatorname{dom} J,\ p^1 \in \partial J(u^1),\ p^2 \in \partial J(u^2)$:

$$\left\langle p^1 - p^2, u^1 - u^2 \right\rangle \geq 0.$$

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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then the set-valued map ∂J is **closed**, i.e. $p^* \in \partial J(u^*)$ whenever

$$\exists (u^k, p^k) \to (u^*, p^*) \in (\operatorname{dom} J) \times \mathbb{E} \ \text{ s.t. } p^k \in \partial J(u^k) \ \ \forall k.$$

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Optimality condition

Theorem

Given any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is

$$0 \in \partial J(u^*).$$

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Constrained optimization as special case

If u^* minimizes $\widetilde{J} = J + \delta_C$ with convex function $J : \mathbb{E} \to \mathbb{R}$ and closed convex subset $C \subset \mathbb{E}$, then $0 \in \partial \widetilde{J}(u^*) \Leftrightarrow$

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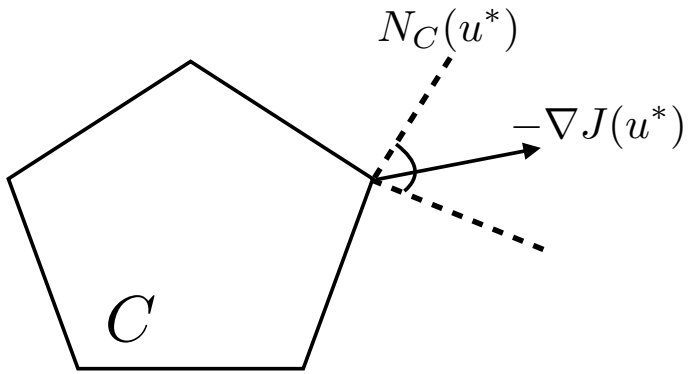

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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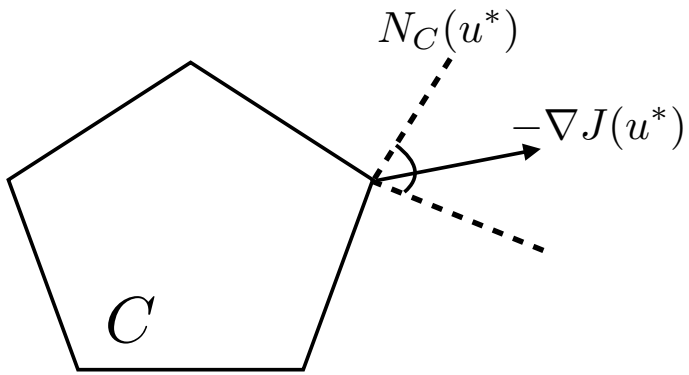

Remark

The optimality condition $0 \in \partial J(u^*) + N_C(u^*)$ is *geometric*. More explicit characterization replies on the *algebraic* representation of $N_C(u^*)$ (e.g., the Karush-Kuhn-Tucker (KKT) conditions) typically under certain *constraint qualifications*.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Example: Normal cone of linear-inequality constraints

Let $C = \{u \in \mathbb{R}^n : Au \leq b\}$ with given $b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$. Then

$$N_C(u) = \{A^\top \lambda : \lambda \geq 0,\ \lambda_i = 0 \text{ whenever } (Au - b)_i < 0\}.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Convex Conjugate

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Convex conjugate

Definition

Given a function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the **convex conjugate** (a.k.a. Legendre-Fenchel transform) of J is defined by

$$J^*(p) = \sup_{u \in \mathbb{E}} \{\langle u, p \rangle - J(u)\} \quad \forall p \in \mathbb{E}.$$

Tao Wu
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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Examples (as exercise)

1. $J(u) = |||u||| \;\Rightarrow\; J^*(p) = \delta\{|||p|||_* \leq 1\}$. $|||\cdot|||_*$ is the *dual norm* of $|||\cdot|||$, i.e., $|||p|||_* = \sup\{\langle p, u \rangle : |||u||| \leq 1\}$.
2. $J(u) = \frac{1}{q}\|u\|_q^q \;\Rightarrow\; J^*(p) = \frac{1}{r}\|p\|_r^r$. $(1 < q < \infty, \frac{1}{q} + \frac{1}{r} = 1)$
3. $J(u) = \sum_{i=1}^n u_i \log u_i + \delta\{u \in \Delta^n\}$ (negative entropy)
   $\Rightarrow J^*(p) = \log(\sum_{i=1}^n \exp(p_i))$. (log-sum-exp)

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Calculus rules

- Scalar multiplication: $\widetilde{J}(\cdot) = \alpha J(\cdot) \;\Rightarrow\; \widetilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$.
- Translation: $\widetilde{J}(\cdot) = J(\cdot - z) \;\Rightarrow\; \widetilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z \rangle$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

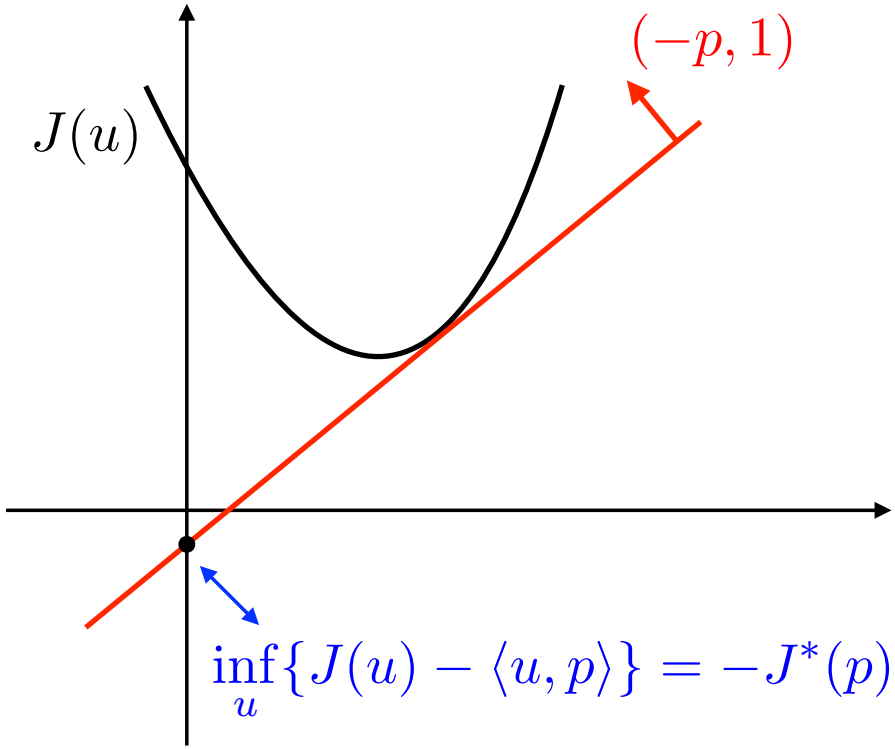
Geometric interpretation

Geometrically, convex conjugation maps

the normal vector of a supporting hyperplane to the epigraph

to

the intersection with the vertical axis.



Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Fenchel-Young inequality, order reversing property

Theorem (Fenchel-Young inequality)

For any function $J : \mathbb{E} \to \overline{\mathbb{R}}$ and $(u, p) \in \mathbb{E} \times \mathbb{E}$, we have

$$J(u) + J^*(p) \geq \langle u, p \rangle .$$

If J is also convex, the equality holds iff $p \in \partial J(u)$ with $(u, p) \in \operatorname{dom} J \times \operatorname{dom} J^*$.

Proof: (i) $J(u) + J^*(p) \geq \langle u, p \rangle$ follows directly from the definition of convex conjugate.

(ii) The equality holds only if $(u, p) \in \operatorname{dom} J \times \operatorname{dom} J^*$. Moreover, $p \in \partial J(u)$ is the sufficient and necessary condition for: $\min_{u \in \mathbb{E}} J(u) - \langle u, p \rangle$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Theorem (order reversing)

For any $J_1, J_2 : \mathbb{E} \to \overline{\mathbb{R}}$, $J_1^*(\cdot) \leq J_2^*(\cdot)$ whenever $J_1(\cdot) \geq J_2(\cdot)$.

Proof: For any (u, p), we have $\langle u, p \rangle - J_1(u) \leq \langle u, p \rangle - J_2(u)$. Taking supremum over u on both sides yields $J_1^*(p) \leq J_2^*(p)$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Biconjugate

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$, and $J^{**} = (J^*)^*$ is the **biconjugate** of J.

In general:

1. $J^{**}(\cdot) \leq J(\cdot)$.
2. J^* is convex and lsc.

If J is proper, convex, and lsc, then:

3. $J^{**}(\cdot) = J(\cdot)$.
4. $p \in \partial J(u)$ iff $u \in \partial J^*(p)$.

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Regularity of J and J^*

Definition

1. $J : \mathbb{E} \to \mathbb{R}$ is μ-**strongly convex** if $\exists \mu > 0$ s.t. $J(\cdot) - \frac{\mu}{2}\|\cdot\|^2$ is convex.
2. $J : \mathbb{E} \to \mathbb{R}$ is L-**Lipschitz differentiable** (a.k.a. L-smooth) if J is differentiable and ∇J is Lipschitz with modulus L.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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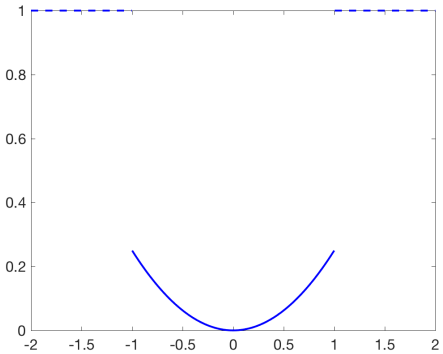
Theorem

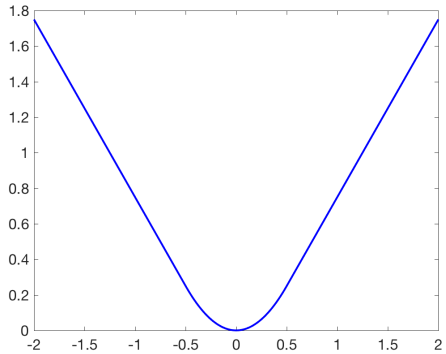
Assume that $J : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, convex, and lsc. Then J is μ-strongly convex iff J^* is $\frac{1}{\mu}$-Lipschitz differentiable.

Proof: on board.

compactly supported quadratic



Huber function



Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Duality Theory

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Fenchel-Rockafellar duality

- Consider the model problem:

$$\inf_{u\in\mathbb{R}^n}\{F(Ku)+G(u)\},$$

with $K \in \mathbb{R}^{m\times n}$, and $F : \mathbb{R}^m \to \overline{\mathbb{R}}$, $G : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper, convex, and lsc functions.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- The **weak duality** ($\mathcal{P}^*\geq\mathcal{D}^*$) always holds:

$$\begin{aligned}\mathcal{P}^* &:= \inf_u\{F(Ku)+G(u)\}\\ &= \inf_u\sup_p\{\langle p,Ku\rangle - F^*(p)+G(u)\}\\ &\geq \sup_p\inf_u\{\langle K^\top p,u\rangle + G(u)-F^*(p)\}\\ &= \sup_p\{-G^*(-K^\top p)-F^*(p)\}=:\mathcal{D}^*.\end{aligned}$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Define the **duality gap**:

$$\mathcal{G}(u,p)=F(Ku)+G(u)+G^*(-K^\top p)+F^*(p).$$

Note that (i) $\mathcal{G}(u,p)\geq 0\ \ \forall(u,p)$; (ii) $\mathcal{G}(u,p)=0$ is an optimality criterion.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- $\mathcal{G}(u^*, p^*) = 0 \Leftrightarrow \mathcal{P}^* = \mathcal{D}^* \Leftrightarrow (u^*, p^*)$ solves the **saddle point problem** with $\mathcal{L}(u, p) := \langle p, Ku \rangle - F^*(p) + G(u)$:

$$\mathcal{L}(u^*, p) \leq \mathcal{L}(u^*, p^*) \leq \mathcal{L}(u, p^*) \quad \forall (u, p).$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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$$\mathcal{L}(u^*, p) \leq \mathcal{L}(u^*, p^*) \leq \mathcal{L}(u, p^*) \quad \forall (u, p).$$

Theorem (Fenchel-Rockafellar duality)

Assume $\exists \bar{u} \in \operatorname{dom} G$ s.t. F is continuous at $K\bar{u}$. Then the **strong duality** holds: $\mathcal{P}^* = \mathcal{D}^*$. Moreover, (u^*, p^*) is the optimal solution pair iff

$$\begin{cases} Ku^* \in \partial F^*(p^*), \\ -K^\top p^* \in \partial G(u^*). \end{cases}$$

Proof: on board.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Example: Total-variation image denoising

- Primal problem (given $\Omega \subset \mathbb{R}^d,\ f \in \mathbb{R}^\Omega,\ \alpha > 0,\ q \in [1, \infty]$):

$$\min_{u \in \mathbb{R}^\Omega} \alpha \|\nabla u\|_{1,q} + \frac{1}{2}\|u - f\|^2.$$

Here $\|p\|_{1,q} = \sum_{j \in \Omega} |p_j|_{\ell^q}$ for each $p \in \mathbb{R}^{|\Omega| \times d}$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Apply the Fenchel-Rockafellar duality with

$$F(\cdot) = \alpha \|\cdot\|_{1,q}, \quad K = \nabla, \quad G(\cdot) = \frac{1}{2}\|\cdot - f\|^2.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Saddle point problem ($1/q + 1/q' = 1$):

$$\max_p \min_u \langle p, \nabla u \rangle - \delta\{\|p\|_{\infty,q'} \le \alpha\} + \frac{1}{2}\|u - f\|^2.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Dual problem:

$$\min_p \frac{1}{2}\|\nabla^\top p\|^2 + \langle \nabla^\top p, f \rangle + \delta\{\|p\|_{\infty,q'} \le \alpha\}.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Proximal Operator

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Proximal operator

Definition

Given a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$, we define the **proximal operator** of J by

$$\text{prox}_{\tau J}(v) = \arg\min_u J(u) + \frac{1}{2\tau}\|u - v\|^2.$$

Observations

1. The minimization in prox always has a unique minimizer.
2. By checking the optimality condition,

   $$u = \text{prox}_{\tau J}(v) \Leftrightarrow 0 \in \tau\partial J(u) + u - v \Leftrightarrow u = (I + \tau\partial J)^{-1}(v).$$

   Thus, $\text{prox}_{\tau J} = (I + \tau\partial J)^{-1}$, a.k.a. the **resolvent** of ∂J.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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   Thus, $\text{prox}_{\tau J} = (I + \tau\partial J)^{-1}$, a.k.a. the **resolvent** of ∂J.
3. u^* is a **fixed point** of $\text{prox}_{\tau J}$, i.e. $u^* = \text{prox}_{\tau J}(u^*)$,
   $\Leftrightarrow 0 \in \partial J(u^*)$, i.e. u^* is a minimizer of J.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Examples of proximal operator

1. Indicator function. Let C be nonempty, closed, convex $\Rightarrow$

$$\text{prox}_{\tau\delta_C}(\cdot) = \text{proj}_C(\cdot).$$

In this case: proximal gradient $\Rightarrow$ projected gradient.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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2. Linear approximation. $\widetilde{J}(\cdot) := J(\bar{u}) + \langle \nabla J(\bar{u}), \cdot - \bar{u} \rangle \Rightarrow$

$$\begin{aligned}\text{prox}_{\tau\widetilde{J}}(\bar{u}) &= \arg\min_u \left\{ \frac{1}{2\tau}\|u - \bar{u}\|^2 + \langle \nabla J(\bar{u}), u - \bar{u} \rangle \right\} \\ &= \bar{u} - \tau \nabla J(\bar{u}). \quad \text{(gradient descent step)}\end{aligned}$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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3. Quadratic approximation.

$\widetilde{J}(\cdot) := J(\bar{u}) + \langle \nabla J(\bar{u}), \cdot - \bar{u} \rangle + \frac{1}{2}\left\langle \nabla^2 J(\bar{u})(\cdot - \bar{u}), \cdot - \bar{u} \right\rangle \Rightarrow$

$$\begin{aligned}\text{prox}_{\tau\widetilde{J}}(\bar{u}) = \arg\min_u \Bigg\{ &\frac{1}{2\tau}\|u-\bar{u}\|^2 + \langle \nabla J(\bar{u}), u-\bar{u}\rangle \\ &+ \frac{1}{2}\left\langle \nabla^2 J(\bar{u})(u-\bar{u}), u-\bar{u} \right\rangle \Bigg\}\end{aligned}$$

$$= \bar{u} - \left(\nabla^2 J(\bar{u}) + \frac{1}{\tau} I\right)^{-1} \nabla J(\bar{u}). \quad \text{(damped Newton step)}$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Proximal operator in optimization algorithms

Proximal gradient

Let us solve the convex optimization:

$$\min_u F(u) + G(u),$$

with G continuously differentiable and F non-differentiable.

The **proximal gradient** iteration appears as:

$$u^{k+1} = \text{prox}_{\tau F}(u^k - \tau \nabla G(u^k)).$$

Derivation of proximal gradient:

$$\begin{aligned}
& u^* \in \arg\min_u F(u) + G(u) \\
\Leftrightarrow\ & 0 \in \partial F(u^*) + \nabla G(u^*) \\
\Leftrightarrow\ & u^* + \tau \partial F(u^*) \ni u^* - \tau \nabla G(u^*) \\
\Leftrightarrow\ & u^* = (I + \tau \partial F)^{-1}(u^* - \tau \nabla G(u^*)) \\
\rightsquigarrow\ & u^{k+1} = \text{prox}_{\tau F}(u^k - \tau \nabla G(u^k)).
\end{aligned}$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Logistic regression (programming exercise)

- Classification of MNIST[1] handwritten digits.
- $C = 10$ classes; grayscale images of pixels $M = 28 \times 28$.
- $N = 60000$ training images $X \in \mathbb{R}^{N \times M}$ with ground-truth labels $Y \in \{1, ..., C\}^N$; 10000 test samples for evaluation.
- Performance by error rate: Convolutional neural network $< 1\%$; **Logistic regression** $< 8\%$.

[1] http://yann.lecun.com/exdb/mnist/

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Logistic regression: Train a *linear classifier* $x \mapsto Wx + b$ with $W \in \mathbb{R}^{C \times M}$ and $b \in \mathbb{R}^C$, which parameterizes *likelihood* via *softmax function*:

$$\mathcal{P}(y|x; W, b) = \frac{\exp(\langle W_{y,.}, x \rangle + b_y)}{\sum_{k=1}^{C} \exp(\langle W_{k,.}, x \rangle + b_k)}.$$

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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Minimize: negative log-likelihood $\mathcal{P}$ + regularization, i.e.,

$$\min_{W,b} \; -\frac{1}{N} \sum_{n=1}^{N} \log \mathcal{P}(Y_n|X_{n,.}; W, b) + \mathcal{R}_W(W) + \mathcal{R}_b(b).$$

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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Moreau identity

Theorem (Moreau identity)

Let $\tau > 0$ and $J : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. Then the following identity holds:

$$\mathrm{id}(\cdot) = \mathrm{prox}_{\tau J}(\cdot) + \tau \, \mathrm{prox}_{\frac{1}{\tau} J^*}(\cdot/\tau).$$

In particular, $\tau = 1 \Rightarrow \mathrm{id}(\cdot) = \mathrm{prox}_{J}(\cdot) + \mathrm{prox}_{J^*}(\cdot)$.

Proof:

$$\begin{aligned}
& v = \tau \, \mathrm{prox}_{\frac{1}{\tau} J^*}(u/\tau) \\
\Leftrightarrow \quad & \left(I + \frac{1}{\tau} \partial J^* \right)^{-1} (u/\tau) = v/\tau \\
\Leftrightarrow \quad & \partial J^*(v/\tau) \ni u - v \\
\Leftrightarrow \quad & v/\tau \in \partial J(u - v) \\
\Leftrightarrow \quad & u - v = (I + \tau \partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u).
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Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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\Leftrightarrow \quad & u - v = (I + \tau \partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u).
\end{aligned}$$

Remark

The Moreau identity suggests that if one of $\mathrm{prox}_{J}(\cdot)$ and $\mathrm{prox}_{J^*}(\cdot)$ is computable, so is the other.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Infimal convolution

Definition

Let $F, G : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. The **infimal convolution** (or inf convolution) of F and G is defined by

$$(F \square G)(u) = \inf_{v \in \mathbb{E}} \{F(u - v) + G(v)\},$$

with $\operatorname{dom}(F \square G) = \operatorname{dom} F + \operatorname{dom} G$.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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with $\operatorname{dom}(F \square G) = \operatorname{dom} F + \operatorname{dom} G$.

Theorem

Let $F, G : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. Then

$$(F \square G)^* = F^* + G^*.$$

Proof: $(F \square G)^*(p) = \sup_{u,v}\{\langle p, u\rangle - F(v) - G(u - v)\} = \sup_{u,v}\{\langle p, v\rangle - F(v) + \langle p, u - v\rangle - G(u - v)\} = F^*(p) + G^*(p)$.

Analogy to integral convolution

By convolution theorem, $\widehat{F * G} = \widehat{F} \cdot \widehat{G}$ where $\widehat{\ }$ denotes the Fourier transform and $*$ the integral convolution.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Moreau envelope

Definition

The **Moreau envelope** of a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is defined for each $u \in \mathbb{E}$ by

$$\begin{aligned} \mathrm{env}_{\tau J}(u) &:= \left(J \square \frac{1}{2\tau} \|\cdot\|^2 \right)(u) \\ &= \inf_{v \in \mathbb{E}} \left\{ J(v) + \frac{1}{2\tau} \|v - u\|^2 \right\} \\ &= J(\mathrm{prox}_{\tau J}(u)) + \frac{1}{2\tau} \|\mathrm{prox}_{\tau J}(u) - u\|^2. \end{aligned}$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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Example

$J : u \mapsto \|u\| \;\Rightarrow\; \mathrm{env}_{\tau J}$ is the *Huber function*:

$$\mathrm{env}_{\tau J}(u) = \begin{cases} \frac{1}{2\tau}\|u\|^2 & \text{if } \|u\| \le \tau, \\ \|u\| - \frac{\tau}{2} & \text{if } \|u\| > \tau. \end{cases}$$

Observation: $\mathrm{env}_{\tau J}$ does smoothing on J.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

Properties of Moreau envelope

- Recall the theorem: $(F \square G)^* = F^* + G^* \Rightarrow$

$$(\mathrm{env}_{\tau J})^* = J^* + \left(\frac{1}{2\tau}\|\cdot\|^2\right)^* = J^* + \frac{\tau}{2}\|\cdot\|^2.$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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- Recall the theorem: J is μ-strongly convex iff J^* is $\frac{1}{\mu}$-Lipschitz differentiable.

  $\Rightarrow$ $\text{env}_{\tau J}$ is $\frac{1}{\tau}$-Lipschitz differentiable.

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator

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  $\Rightarrow \mathrm{env}_{\tau J}$ is $\frac{1}{\tau}$-Lipschitz differentiable.

- $\nabla \mathrm{env}_{\tau J}$ can be calculated as:

$$\begin{aligned}
p = \nabla \mathrm{env}_{\tau J}(u) &\Leftrightarrow u \in \partial(\mathrm{env}_{\tau J})^*(p) = \partial J^*(p) + \tau p \\
&\Leftrightarrow u - \tau p \in \partial J^*(p) \Leftrightarrow \partial J(u - \tau p) \ni p \\
&\Leftrightarrow \tau \partial J(u - \tau p) \ni \tau p \Leftrightarrow (I + \tau \partial J)(u - \tau p) \ni u \\
&\Leftrightarrow u - \tau p = (I + \tau \partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u) \\
&\Leftrightarrow \nabla \mathrm{env}_{\tau J}(u) = p = \frac{1}{\tau}(u - \mathrm{prox}_{\tau J}(u)).
\end{aligned}$$

  Hence,

$$\mathrm{prox}_{\tau J}(u) = u - \tau \nabla \mathrm{env}_{\tau J}(u).$$

Convex Set
Convex Function
Existence of Minimizer
Subdifferential
Convex Conjugate
Duality Theory
Proximal Operator