Proof Script for WS2019/20 Convex Optimization*

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1 Convex Analysis

Theorem 1.1 (separation of convex sets)**.** *Let C_1, C_2 be nonempty convex subsets of $\mathbb{E}$.*

1. *Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E}\backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.*
$$\langle v, w\rangle > \alpha \geq \langle v, u\rangle\,, \quad \forall u \in C_1.$$
2. *Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E}\backslash C_1$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.*
$$\langle v, w\rangle \geq \alpha \geq \langle v, u\rangle\,, \quad \forall u \in C_1.$$
3. *Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.*
$$\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$
4. *Assume $\emptyset \neq \operatorname{int} C_1 \subset \mathbb{E}\backslash C_2$. Then $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.*
$$\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

Proof. (1) Consider the projection of w onto C_1, i.e., set $u^* := \arg\min_{u \in C_1} \frac{1}{2}\|u - w\|^2$ or, equivalently via the variational inequality: $\langle u - u^*, u^* - w\rangle \geq 0\ \forall u \in C_1$. Now let $v := w - u^* \neq 0$. Then $\forall u \in C_1$, we have $\langle v, w\rangle = \langle w - u^*, w\rangle = \|w - u^*\|^2 + \langle w - u^*, u^*\rangle \geq \|w - u^*\|^2 + \langle w - u^*, u\rangle = \|v\|^2 + \langle v, u\rangle$. Set $\alpha := \sup\{\langle v, u\rangle : u \in C_1\}$. Note $\alpha < \infty$ since $\langle v, u\rangle \leq \langle v, u^*\rangle\ \forall u \in C$. Thus $\langle v, w\rangle > \langle v, w\rangle - \|v\|^2 \geq \alpha \geq \langle v, u\rangle\ \forall u \in C_1$.

(2) Since $\mathbb{E}\backslash C_1$ is closed, $\exists w^k \in \mathbb{E}\backslash \operatorname{cl} C_1$ s.t. $w^k \to w$. For each w^k, by (i), $\exists v^k \in \mathbb{E}$ with $\|v^k\| \equiv 1$ s.t. $\left\langle v^k, w^k\right\rangle \leq \left\langle v^k, u\right\rangle\ \forall u \in \operatorname{cl} C_1$. Hence $v^k \to v \in \mathbb{E}$ along a subsequence s.t. $\|v\| = 1$ and $\alpha := \langle v, w\rangle \leq \langle v, u\rangle\ \forall u \in C_1 \subset \operatorname{cl} C_1$.

(3) Let $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1,\ u^2 \in C_2\}$. Note that C is a convex, open set, and $0 \notin C$. By (2), $\exists v \in \mathbb{E}$ with $\|v\| = 1$ s.t. $\left\langle -v, u^2 - u^1\right\rangle \geq \langle -v, 0\rangle = 0$ or, equivalently, $\left\langle v, u^1\right\rangle \geq \left\langle v, u^2\right\rangle\ \forall u^1 \in C_1,\ u^2 \in C_2$. Set $\alpha := \sup\{\left\langle v, u^2\right\rangle : u^2 \in C_2\}$, then we conclude that $\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle\ \forall u^1 \in C_1,\ u^2 \in C_2$.

(4) By applying (3) to $\operatorname{int} C_1$ and C_2, we have $\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle\ \forall u^1 \in \operatorname{int} C_1,\ u^2 \in C_2$. The inequality remains true for all $u_1 \in C_1$. □

*Please report typos to: tao.wu@tum.de