

# Proof Script for WS2019/20 Convex Optimization\*

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## 1 Convex Analysis

**Theorem 1.1** (separation of convex sets). *Let  $C_1, C_2$  be nonempty convex subsets of  $\mathbb{E}$ .*

1. *Assume  $C_1$  is closed and  $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$ . Then  $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$  s.t.*

$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

2. *Assume  $C_1$  is open and  $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$ . Then  $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$  s.t.*

$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

3. *Assume  $C_1 \cap C_2 = \emptyset$  and  $C_1$  is open. Then  $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$  s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

4. *Assume  $\emptyset \neq \text{int } C_1 \subset \mathbb{E} \setminus C_2$ . Then  $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$  s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

*Proof.* (1) Consider the projection of  $w$  onto  $C_1$ , i.e., set  $u^* := \arg \min_{u \in C_1} \frac{1}{2} \|u - w\|^2$  or, equivalently via the variational inequality:  $\langle u - u^*, u^* - w \rangle \geq 0 \forall u \in C_1$ . Now let  $v := w - u^* \neq 0$ . Then  $\forall u \in C_1$ , we have  $\langle v, w \rangle = \langle w - u^*, w \rangle = \|w - u^*\|^2 + \langle w - u^*, u^* \rangle \geq \|w - u^*\|^2 + \langle w - u^*, u \rangle = \|v\|^2 + \langle v, u \rangle$ . Set  $\alpha := \sup\{\langle v, u \rangle : u \in C_1\}$ . Note  $\alpha < \infty$  since  $\langle v, u \rangle \leq \langle v, u^* \rangle \forall u \in C$ . Thus  $\langle v, w \rangle > \langle v, w \rangle - \|v\|^2 \geq \alpha \geq \langle v, u \rangle \forall u \in C_1$ .

(2) Since  $\text{cl } C_1$  is closed,  $\exists w^k \in \mathbb{E} \setminus \text{cl } C_1$  s.t.  $w^k \rightarrow w$ . For each  $w^k$ , by (i),  $\exists v^k \in \mathbb{E}$  with  $\|v^k\| \equiv 1$  s.t.  $\langle v^k, w^k \rangle \leq \langle v^k, u \rangle \forall u \in \text{cl } C_1$ . Hence  $v^k \rightarrow v \in \mathbb{E}$  along a subsequence s.t.  $\|v\| = 1$  and  $\alpha := \langle v, w \rangle \leq \langle v, u \rangle \forall u \in C_1 \subset \text{cl } C_1$ .

(3) Let  $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1, u^2 \in C_2\}$ . Note that  $C$  is a convex, open set, and  $0 \notin C$ . By (2),  $\exists v \in \mathbb{E}$  with  $\|v\| = 1$  s.t.  $\langle -v, u^2 - u^1 \rangle \geq \langle -v, 0 \rangle = 0$  or, equivalently,  $\langle v, u^1 \rangle \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$ . Set  $\alpha := \sup\{\langle v, u^2 \rangle : u^2 \in C_2\}$ , then we conclude that  $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$ .

(4) By applying (3) to  $\text{int } C_1$  and  $C_2$ , we have  $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in \text{int } C_1, u^2 \in C_2$ . The inequality remains true for all  $u_1 \in C_1$ .  $\square$

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**Theorem 1.2.** *A proper convex function  $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$  is locally Lipschitz at any  $u \in \text{rint dom } J$ .*

*Proof.* Without loss of generality, we restrict  $J : \text{aff dom } J \rightarrow \overline{\mathbb{R}}$ .

(i) Claim: If  $M = \sup\{J(v) : v \in B_\epsilon(u)\} < \infty$  with  $\epsilon > 0$ , then  $J$  is locally Lipschitz at  $u$ .

First, by convexity of  $J$  we have  $\forall v \in B_\epsilon(u) : J(v) \geq 2J(u) - J(2u - v) \geq 2J(u) - M$ . Thus,  $\sup\{|J(v)| : v \in B_\epsilon(u)\} \leq M + 2|J(u)|$ .

Next, we show  $J$  is Lipschitz on  $B_{\epsilon/2}(u)$ . Let  $v, w \in B_{\epsilon/2}(u)$  be given. Take  $z \in B_\epsilon(u)$  s.t.  $w = (1 - t)v + tz$  for some  $t \in [0, 1]$  and  $\|z - v\| \geq \epsilon/2$ . By convexity,  $J(w) - J(v) \leq t(J(z) - J(v)) \leq 2t(M - J(u))$ . Since  $t(z - v) = w - v$ , we have  $t = \|w - v\|/\|z - v\| \leq 2\|w - v\|/\epsilon$  and  $J(w) - J(v) \leq (4(M - J(u))/\epsilon)\|w - v\|$ . Analogously, one can show  $J(v) - J(w) \leq (4(M - J(u))/\epsilon)\|w - v\|$ . Hence,  $J$  is Lipschitz on  $B_{\epsilon/2}(u)$  with modulus  $4(M - J(u))/\epsilon$ .

(ii) Let  $u \in \text{rint dom } J$  and  $n = \dim(\text{aff dom } J)$ . Then by Carathéodory's theorem,  $\exists\{\alpha^i\}_{i=1}^{n+1} \subset (0, 1)$ ,  $\{u^i\}_{i=1}^{n+1} \subset \text{dom } J$  s.t.  $u = \sum_{i=1}^{n+1} \alpha^i u^i$ ,  $\sum_{i=1}^{n+1} \alpha^i = 1$ , i.e.,  $u$  belongs to the relative interior of the convex hull of  $\{u^i\}_{i=1}^{n+1}$ . Thus one can apply (i) to assert that  $J$  is locally Lipschitz at  $u$ .  $\square$

**Theorem 1.3.** *For any proper convex function  $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ , if  $u^* \in \text{dom } J$  is a local minimizer of  $J$ , then it is also a global minimizer.*

*Proof.* By the definition of a local minimizer,  $\exists\epsilon > 0$  s.t.  $J(u^*) \leq J(u) \forall u \in B_\epsilon(u^*)$ . For the sake of contradiction, assume  $\exists\bar{u} \in \mathbb{E}$  s.t.  $J(\bar{u}) < J(u^*)$ . By convexity of  $J$ , we have  $J(\alpha\bar{u} + (1 - \alpha)u^*) \leq J(u^*) - \alpha(J(u^*) - J(\bar{u})) < J(u^*) \forall \alpha \in (0, 1]$ . This violates the local optimality of  $u^*$  as  $\alpha \rightarrow 0^+$ .  $\square$

**Theorem 1.4.** *Any proper function  $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ , which is bounded from below, coercive, and lsc, has a (global) minimizer.*

*Proof.* Let  $\{u^k\}$  be an infimizing sequence for  $J$ , i.e.,  $\lim_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u) \in \mathbb{R}$ . Since  $\{J(u^k)\}$  is uniformly bounded from above, by coercivity of  $J$  we have  $\{u^k\}$  uniformly bounded. By compactness,  $u^k \rightarrow u^*$  along a subsequence. Since  $J$  is lsc, we have  $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u)$ , which implies  $J(u^*) = \inf_{u \in \mathbb{E}} J(u)$  or  $u^*$  is a minimizer of  $J$ .  $\square$

**Theorem 1.5.** *The minimizer of a strictly convex function  $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$  is unique.*

*Proof.* Let  $u, v \in \mathbb{E}$  be two (global) minimizers s.t.  $u \neq v$  and  $J(u) = J(v) = J^*$ . By strict convexity of  $J$ ,  $J(\alpha u + (1 - \alpha)v) < \alpha J(u) + (1 - \alpha)J(v) = J^*$  for all  $\alpha \in (0, 1)$ , which contradicts the global optimality of  $u$  and  $v$ .  $\square$