# Proof Script for WS2019/20 Convex Optimization*

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## 1 Convex Analysis

**Theorem 1.1** (separation of convex sets)**.** *Let $C_1$, $C_2$ be nonempty convex subsets of $\mathbb{E}$.*

1. *Assume $C_1$ is closed and $C_2 = \{w\} \subset \mathbb{E}\backslash C_1$. Then* $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w\rangle > \alpha \geq \langle v, u\rangle, \quad \forall u \in C_1.$$
2. *Assume $C_1$ is open and $C_2 = \{w\} \subset \mathbb{E}\backslash C_1$. Then* $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\langle v, w\rangle \geq \alpha \geq \langle v, u\rangle, \quad \forall u \in C_1.$$
3. *Assume $C_1 \cap C_2 = \emptyset$ and $C_1$ is open. Then* $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$
4. *Assume $\emptyset \neq \operatorname{int} C_1 \subset \mathbb{E}\backslash C_2$. Then* $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.
$$\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

*Proof.* (1) Consider the projection of $w$ onto $C_1$, i.e., set $u^* := \arg\min_{u \in C_1} \frac{1}{2}\|u - w\|^2$ or, equivalently via the variational inequality: $\langle u - u^*, u^* - w\rangle \geq 0\ \forall u \in C_1$. Now let $v := w - u^* \neq 0$. Then $\forall u \in C_1$, we have $\langle v, w\rangle = \langle w - u^*, w\rangle = \|w - u^*\|^2 + \langle w - u^*, u^*\rangle \geq \|w - u^*\|^2 + \langle w - u^*, u\rangle = \|v\|^2 + \langle v, u\rangle$. Set $\alpha := \sup\{\langle v, u\rangle : u \in C_1\}$. Note $\alpha < \infty$ since $\langle v, u\rangle \leq \langle v, u^*\rangle\ \forall u \in C$. Thus $\langle v, w\rangle > \langle v, w\rangle - \|v\|^2 \geq \alpha \geq \langle v, u\rangle\ \forall u \in C_1$.

(2) We have either $w \in \mathbb{E}\backslash \operatorname{cl} C_1$ or $w \in \operatorname{bdry} C_1$. In either case, $\exists w^k \in \mathbb{E}\backslash \operatorname{cl} C_1$ s.t. $w^k \to w$. For each $w^k$, by (i), $\exists v^k \in \mathbb{E}$ with $\|v^k\| \equiv 1$ s.t. $\left\langle v^k, w^k\right\rangle \leq \left\langle v^k, u\right\rangle\ \forall u \in \operatorname{cl} C_1$. Hence $v^k \to v \in \mathbb{E}$ along a subsequence s.t. $\|v\| = 1$ and $\alpha := \langle v, w\rangle \leq \langle v, u\rangle\ \forall u \in C_1 \subset \operatorname{cl} C_1$.

(3) Let $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1,\ u^2 \in C_2\}$. Note that $C$ is a convex, open set, and $0 \notin C$. By (2), $\exists v \in \mathbb{E}$ with $\|v\| = 1$ s.t. $\left\langle -v, u^2 - u^1\right\rangle \geq \langle -v, 0\rangle = 0$ or, equivalently, $\left\langle v, u^1\right\rangle \geq \left\langle v, u^2\right\rangle\ \forall u^1 \in C_1,\ u^2 \in C_2$. Set $\alpha := \sup\{\left\langle v, u^2\right\rangle : u^2 \in C_2\}$, then we conclude that $\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle\ \forall u^1 \in C_1,\ u^2 \in C_2$.

(4) By applying (3) to $\operatorname{int} C_1$ and $C_2$, we have $\left\langle v, u^1\right\rangle \geq \alpha \geq \left\langle v, u^2\right\rangle\ \forall u^1 \in \operatorname{int} C_1,\ u^2 \in C_2$. The inequality remains true for all $u_1 \in C_1$. ☐

*Please report typos to: tao.wu@tum.de

**Theorem 1.2.** *A proper convex function* $J : \mathbb{E} \to \overline{\mathbb{R}}$ *is locally Lipschitz at any* $u \in \operatorname{rint}\operatorname{dom} J$.

*Proof.* Without loss of generality, we restrict $J : \operatorname{aff}\operatorname{dom} J \to \overline{\mathbb{R}}$.

(i) Claim: If $M = \sup\{J(v) : v \in B_\epsilon(u)\} < \infty$ with $\epsilon > 0$, then $J$ is locally Lipschitz at $u$.

First, by convexity of $J$ we have $\forall v \in B_\epsilon(u) : J(v) \geq 2J(u) - J(2u - v) \geq 2J(u) - M$. Thus, $\sup\{|J(v)| : v \in B_\epsilon(u)\} \leq M + 2|J(u)|$.

Next, we show $J$ is Lipschitz on $B_{\epsilon/2}(u)$. Let $v, w \in B_{\epsilon/2}(u)$ be given. Take $z \in B_\epsilon(u)$ s.t. $w = (1 - t)v + tz$ for some $t \in [0, 1]$ and $\|z - v\| \geq \epsilon/2$. By convexity, $J(w) - J(v) \leq t(J(z) - J(v)) \leq 2t(M - J(u))$. Since $t(z - v) = w - v$, we have $t = \|w - v\|/\|z - v\| \leq 2\|w - v\|/\epsilon$ and $J(w) - J(v) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Analogously, one can show $J(v) - J(w) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Hence, $J$ is Lipschitz on $B_{\epsilon/2}(u)$ with modulus $4(M - J(u))/\epsilon$.

(ii) Let $u \in \operatorname{rint}\operatorname{dom} J$ and $n = \dim(\operatorname{aff}\operatorname{dom} J)$. Then by Carathéodory's theorem, $\exists\{\alpha^i\}_{i=1}^{n+1} \subset (0, 1)$, $\{u^i\}_{i=1}^{n+1} \subset \operatorname{dom} J$ s.t. $u = \sum_{i=1}^{n+1} \alpha^i u^i$, $\sum_{i=1}^{n+1} \alpha^i = 1$, i.e., $u$ belongs to the relative interior of the convex hull of $\{u^i\}_{i=1}^{n+1}$. Thus one can apply (i) to assert that $J$ is locally Lipschitz at $u$. □

**Theorem 1.3.** *For any proper convex function* $J : \mathbb{E} \to \overline{\mathbb{R}}$, *if* $u^* \in \operatorname{dom} J$ *is a local minimizer of* $J$, *then it is also a global minimizer.*

*Proof.* By the definition of a local minimizer, $\exists \epsilon > 0$ s.t. $J(u^*) \leq J(u)\ \forall u \in B_\epsilon(u^*)$. For the sake of contradiction, assume $\exists \bar{u} \in \mathbb{E}$ s.t. $J(\bar{u}) < J(u^*)$. By convexity of $J$, we have $J(\alpha\bar{u} + (1 - \alpha)u^*) \leq J(u^*) - \alpha(J(u^*) - J(\bar{u})) < J(u^*)\ \forall \alpha \in (0, 1]$. This violates the local optimality of $u^*$ as $\alpha \to 0^+$. □

**Theorem 1.4.** *Any proper function* $J : \mathbb{E} \to \overline{\mathbb{R}}$, *which is bounded from below, coercive, and lsc, has a (global) minimizer.*

*Proof.* Let $\{u^k\}$ be an infimizing sequence for $J$, i.e., $\lim_{k\to\infty} J(u^k) = \inf_{u\in\mathbb{E}} J(u) \in \mathbb{R}$. Since $\{J(u^k)\}$ is uniformly bounded from above, by coercivity of $J$ we have $\{u^k\}$ uniformly bounded. By compactness, $u^k \to u^*$ along a subsequence. Since $J$ is lsc, we have $J(u^*) \leq \liminf_{k\to\infty} J(u^k) = \inf_{u\in\mathbb{E}} J(u)$, which implies $J(u^*) = \inf_{u\in\mathbb{E}} J(u)$ or $u^*$ is a minimizer of $J$. □

**Theorem 1.5.** *The minimizer of a strictly convex function* $J : \mathbb{E} \to \overline{\mathbb{R}}$ *is unique.*

*Proof.* Let $u, v \in \mathbb{E}$ be two (global) minimizers s.t. $u \neq v$ and $J(u) = J(v) = J^*$. By strict convexity of $J$, $J(\alpha u + (1 - \alpha)v) < \alpha J(u) + (1 - \alpha)J(v) = J^*$ for all $\alpha \in (0, 1)$, which contradicts the global optimality of $u$ and $v$. □