

Proof Script for WS2019/20 Convex Optimization*

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1 Convex Analysis

Theorem 1.1 (separation of convex sets). *Let C_1, C_2 be nonempty convex subsets of \mathbb{E} .*

1. *Assume C_1 is closed and $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, w \rangle > \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

2. *Assume C_1 is open and $C_2 = \{w\} \subset \mathbb{E} \setminus C_1$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, w \rangle \geq \alpha \geq \langle v, u \rangle, \quad \forall u \in C_1.$$

3. *Assume $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

4. *Assume $\emptyset \neq \text{int } C_1 \subset \mathbb{E} \setminus C_2$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t.*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

Proof. (1) Consider the projection of w onto C_1 , i.e., set $u^* := \arg \min_{u \in C_1} \frac{1}{2} \|u - w\|^2$ or, equivalently via the variational inequality: $\langle u - u^*, u^* - w \rangle \geq 0 \forall u \in C_1$. Now let $v := w - u^* \neq 0$. Then $\forall u \in C_1$, we have $\langle v, w \rangle = \langle w - u^*, w \rangle = \|w - u^*\|^2 + \langle w - u^*, u^* \rangle \geq \|w - u^*\|^2 + \langle w - u^*, u \rangle = \|v\|^2 + \langle v, u \rangle$. Set $\alpha := \sup\{\langle v, u \rangle : u \in C_1\}$. Note $\alpha < \infty$ since $\langle v, u \rangle \leq \langle v, u^* \rangle \forall u \in C$. Thus $\langle v, w \rangle > \langle v, w \rangle - \|v\|^2 \geq \alpha \geq \langle v, u \rangle \forall u \in C_1$.

(2) We have either $w \in \mathbb{E} \setminus \text{cl } C_1$ or $w \in \text{bdry } C_1$. In either case, $\exists w^k \in \mathbb{E} \setminus \text{cl } C_1$ s.t. $w^k \rightarrow w$. For each w^k , by (i), $\exists v^k \in \mathbb{E}$ with $\|v^k\| \equiv 1$ s.t. $\langle v^k, w^k \rangle \leq \langle v^k, u \rangle \forall u \in \text{cl } C_1$. Hence $v^k \rightarrow v \in \mathbb{E}$ along a subsequence s.t. $\|v\| = 1$ and $\alpha := \langle v, w \rangle \leq \langle v, u \rangle \forall u \in C_1 \subset \text{cl } C_1$.

(3) Let $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1, u^2 \in C_2\}$. Note that C is a convex, open set, and $0 \notin C$. By (2), $\exists v \in \mathbb{E}$ with $\|v\| = 1$ s.t. $\langle -v, u^2 - u^1 \rangle \geq \langle -v, 0 \rangle = 0$ or, equivalently, $\langle v, u^1 \rangle \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$. Set $\alpha := \sup\{\langle v, u^2 \rangle : u^2 \in C_2\}$, then we conclude that $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$.

(4) By applying (3) to $\text{int } C_1$ and C_2 , we have $\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle \forall u^1 \in \text{int } C_1, u^2 \in C_2$. The inequality remains true for all $u_1 \in C_1$. \square

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Theorem 1.2. *A proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \text{rint dom } J$.*

Proof. Without loss of generality, we restrict $J : \text{aff dom } J \rightarrow \overline{\mathbb{R}}$.

(i) Claim: If $M = \sup\{J(v) : v \in B_\epsilon(u)\} < \infty$ with $\epsilon > 0$, then J is locally Lipschitz at u .

First, by convexity of J we have $\forall v \in B_\epsilon(u) : J(v) \geq 2J(u) - J(2u - v) \geq 2J(u) - M$. Thus, $\sup\{|J(v)| : v \in B_\epsilon(u)\} \leq M + 2|J(u)|$.

Next, we show J is Lipschitz on $B_{\epsilon/2}(u)$. Let $v, w \in B_{\epsilon/2}(u)$ be given. Take $z \in B_\epsilon(u)$ s.t. $w = (1 - t)v + tz$ for some $t \in [0, 1]$ and $\|z - v\| \geq \epsilon/2$. By convexity, $J(w) - J(v) \leq t(J(z) - J(v)) \leq 2t(M - J(u))$. Since $t(z - v) = w - v$, we have $t = \|w - v\|/\|z - v\| \leq 2\|w - v\|/\epsilon$ and $J(w) - J(v) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Analogously, one can show $J(v) - J(w) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Hence, J is Lipschitz on $B_{\epsilon/2}(u)$ with modulus $4(M - J(u))/\epsilon$.

(ii) Let $u \in \text{rint dom } J$ and $n = \dim(\text{aff dom } J)$. Then by Carathéodory's theorem, $\exists\{\alpha^i\}_{i=1}^{n+1} \subset (0, 1)$, $\{u^i\}_{i=1}^{n+1} \subset \text{dom } J$ s.t. $u = \sum_{i=1}^{n+1} \alpha^i u^i$, $\sum_{i=1}^{n+1} \alpha^i = 1$, i.e., u belongs to the relative interior of the convex hull of $\{u^i\}_{i=1}^{n+1}$. Thus one can apply (i) to assert that J is locally Lipschitz at u . \square

Theorem 1.3. *For any proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, if $u^* \in \text{dom } J$ is a local minimizer of J , then it is also a global minimizer.*

Proof. By the definition of a local minimizer, $\exists\epsilon > 0$ s.t. $J(u^*) \leq J(u) \forall u \in B_\epsilon(u^*)$. For the sake of contradiction, assume $\exists\bar{u} \in \mathbb{E}$ s.t. $J(\bar{u}) < J(u^*)$. By convexity of J , we have $J(\alpha\bar{u} + (1 - \alpha)u^*) \leq J(u^*) - \alpha(J(u^*) - J(\bar{u})) < J(u^*) \forall \alpha \in (0, 1]$. This violates the local optimality of u^* as $\alpha \rightarrow 0^+$. \square

Theorem 1.4. *Any proper function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc, has a (global) minimizer.*

Proof. Let $\{u^k\}$ be an infimizing sequence for J , i.e., $\lim_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u) \in \mathbb{R}$. Since $\{J(u^k)\}$ is uniformly bounded from above, by coercivity of J we have $\{u^k\}$ uniformly bounded. By compactness, $u^k \rightarrow u^*$ along a subsequence. Since J is lsc, we have $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u)$, which implies $J(u^*) = \inf_{u \in \mathbb{E}} J(u)$ or u^* is a minimizer of J . \square

Theorem 1.5. *The minimizer of a strictly convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is unique.*

Proof. Let $u, v \in \mathbb{E}$ be two (global) minimizers s.t. $u \neq v$ and $J(u) = J(v) = J^*$. By strict convexity of J , $J(\alpha u + (1 - \alpha)v) < \alpha J(u) + (1 - \alpha)J(v) = J^*$ for all $\alpha \in (0, 1)$, which contradicts the global optimality of u and v . \square

Theorem 1.6. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \text{int dom } J$, $\partial J(u)$ is a nonempty, compact, and convex subset.*

Proof. (i) nonemptiness. Since $(u, J(u)) \notin \text{int epi } J$, by Theorem 1.1 we have $\exists(p, -\alpha) \in \mathbb{E} \times \mathbb{R}$ s.t. $(p, -\alpha) \neq (0, 0)$, $\alpha \geq 0$ by our choice, and $\langle (p, -\alpha), (u - v, J(u) - J(v)) \rangle \geq 0 \forall v \in \text{dom } J$. In fact, we must have $\alpha > 0$ since otherwise $p = 0$. Thus, we conclude that $p/\alpha \in \partial J(u)$.

(ii) boundedness. By Theorem 1.2, J is locally Lipschitz at u with modulus L_u . Let $p \in \partial J(u)$ be fixed. For any $h \in (\text{dom } J) - u$ whenever $\|h\|$ is sufficiently small, we have $\langle p, h \rangle \leq J(u + h) - J(u) \leq L_u \|h\|$. This holds true only if $\|p\| \leq L_u$, which implies boundedness of $\partial J(u)$.

(iii) closedness. Let $v \in \mathbb{E}$ be arbitrarily fixed and $p^k \rightarrow p^*$ where each $p^k \in \partial J(u)$. Then $\forall k : J(v) - J(u) \geq \langle p^k, v - u \rangle$. By continuity, $J(v) - J(u) \geq \langle p^*, v - u \rangle$ when passing $k \rightarrow \infty$. Since v can be arbitrary, we assert $p^* \in \partial J(u)$.

(iv) convexity. Let $v \in \mathbb{E}$ be arbitrarily fixed, and $p, q \in \partial J(u)$. Then we have

$$\begin{aligned} J(v) &\geq J(u) + \langle p, v - u \rangle, \\ J(v) &\geq J(u) + \langle q, v - u \rangle. \end{aligned}$$

Hence, $\forall 0 \leq \alpha \leq 1 : J(v) \geq J(u) + \langle \alpha p + (1 - \alpha)q, v - u \rangle$, i.e., $\alpha p + (1 - \alpha)q \in \partial J(u)$. \square

Theorem 1.7. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a convex function. Then ∂J is a monotone operator, i.e. $\forall u^1, u^2 \in \text{dom } J$, $p^1 \in \partial J(u^1)$, $p^2 \in \partial J(u^2)$:*

$$\langle p^1 - p^2, u^1 - u^2 \rangle \geq 0.$$

Proof. By applying the definition of subdifferential at arbitrarily given $u^1, u^2 \in \text{dom } J$, we have

$$\begin{aligned} J(u^2) &\geq J(u^1) + \langle p^1, u^2 - u^1 \rangle, \\ J(u^1) &\geq J(u^2) + \langle p^2, u^1 - u^2 \rangle. \end{aligned}$$

Adding the two inequalities yields $\langle p^1 - p^2, u^1 - u^2 \rangle \geq 0$. \square

Theorem 1.8. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then ∂J is a closed set-valued map, i.e., $p^* \in \partial J(u^*)$ whenever*

$$\exists (u^k, p^k) \rightarrow (u^*, p^*) \in (\text{dom } J) \times \mathbb{E} \text{ s.t. } p^k \in \partial J(u^k) \forall k.$$

Proof. Let $v \in \mathbb{E}$ be arbitrarily fixed. For each k , $p^k \in \partial J(u^k) \Rightarrow J(v) \geq J(u^k) + \langle p^k, v - u^k \rangle$. Passing $k \rightarrow \infty$, we have $\langle p^k, v - u^k \rangle \rightarrow \langle p^*, v - u^* \rangle$ and $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k)$. Hence, $J(u^*) + \langle p^*, v - u^* \rangle \leq \liminf_{k \rightarrow \infty} \{J(u^k) + \langle p^k, v - u^k \rangle\} \leq J(v)$. Since v can be arbitrary, $p^* \in \partial J(u^*)$. \square

Theorem 1.9. *Given any proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is: $0 \in \partial J(u^*)$.*

Proof. (i) sufficiency. $0 \in \partial J(u^*) \Rightarrow J(u) \geq J(u^*) + \langle 0, u - u^* \rangle = J(u^*) \forall u \in \mathbb{E}$.

(ii) necessity. $J(u^*) \leq J(u) \forall u \in \mathbb{E} \Rightarrow J(u^*) + \langle 0, u - u^* \rangle \leq J(u) \forall u \Rightarrow 0 \in \partial J(u^*)$. \square

Theorem 1.10 (Fenchel-Young inequality). *For any function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ and $(u, p) \in \mathbb{E} \times \mathbb{E}$, we have*

$$J(u) + J^*(p) \geq \langle u, p \rangle.$$

If J is also convex, the equality holds iff $p \in \partial J(u)$ with $(u, p) \in \text{dom } J \times \text{dom } J^$.*

Proof. (i) $J(u) + J^*(p) \geq \langle u, p \rangle$ follows directly from the definition of convex conjugate. (ii) The equality holds only if $(u, p) \in \text{dom } J \times \text{dom } J^*$. Moreover, $p \in \partial J(u)$ is the sufficient and necessary condition for $\min_{u \in \mathbb{E}} \{J(u) - \langle u, p \rangle\}$. \square

Theorem 1.11 (order reversing). *For any $J_1, J_2 : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, we have $J_1^*(\cdot) \leq J_2^*(\cdot)$ whenever $J_1(\cdot) \geq J_2(\cdot)$.*

Proof. Given any (u, p) , we have $\langle u, p \rangle - J_1(u) \leq \langle u, p \rangle - J_2(u)$. Taking supremum over u on both sides yields $J_1^*(p) \leq J_2^*(p)$. \square

Theorem 1.12. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, and $J^{**} = (J^*)^*$ be the biconjugate of J . In general:*

1. $J^{**}(\cdot) \leq J(\cdot)$.

2. J^* is convex and lsc.

If J is proper, convex, and lsc, then:

3. $J^{**}(\cdot) = J(\cdot)$.

4. $p \in \partial J(u)$ iff $u \in \partial J^*(p)$.

Proof. (1) Since $J^{**}(u) = \sup_p \{\langle p, u \rangle - J^*(p)\}$ and, by Theorem 1.10, $\langle p, u \rangle - J^*(p) \leq J(u) \forall p$, we have $J^{**}(u) \leq J(u)$ for arbitrarily given u .

(2) (i) convexity. Let $p, q \in \mathbb{E}$, $0 \leq \alpha \leq 1$. Then $J^*(\alpha p + (1-\alpha)q) = \sup_u \{\langle u, \alpha p + (1-\alpha)q \rangle - J(u)\} \leq \sup_u \{\langle \alpha u, p \rangle - \alpha J(u)\} + \sup_u \{\langle (1-\alpha)u, q \rangle - (1-\alpha)J(u)\} = \alpha J^*(p) + (1-\alpha)J^*(q)$.

(ii) lsc. Note $\text{epi } J^* = \{(p, \alpha) \in \mathbb{E} \times \mathbb{R} : \langle u, p \rangle - J(u) \leq \alpha \forall u\} = \bigcap_u \text{epi } \Phi_u$ where $\Phi_u(\cdot) = \langle u, \cdot \rangle - J(u)$. Since each $\text{epi } \Phi_u$ and any arbitrary intersection of closed sets is closed, $\text{epi } J^*$ is closed and hence J^* is lsc.

(3) For the sake of contradiction, assume $\exists \bar{u} \in \text{dom } J^{**}$ s.t. $J(\bar{u}) > J^{**}(\bar{u})$. Let $0 < d < J(\bar{u}) - J^{**}(\bar{u})$ be fixed. Since $(\bar{u}, J(\bar{u}) - d) \notin \text{epi } J$ and $\text{epi } J$ is convex and closed, by Theorem 1.1, $\exists (\bar{p}, -1) \in \mathbb{E} \times \mathbb{R}$ s.t. $\langle (\bar{p}, -1), (\bar{u}, J(\bar{u}) - d) \rangle \geq \langle (\bar{p}, -1), (u, \alpha) \rangle \forall (u, \alpha) \in \text{epi } J$. In particular, $\langle \bar{p}, \bar{u} \rangle - J(\bar{u}) + d \geq \langle \bar{p}, u \rangle - J(u) \forall u \in \text{dom } J$. Hence, $\langle \bar{p}, \bar{u} \rangle - J(\bar{u}) + d \geq J^*(\bar{p}) \geq \langle \bar{p}, \bar{u} \rangle - J^{**}(\bar{u})$ by Theorem 1.10. Thus we have $J^{**}(\bar{u}) + d \geq J(\bar{u})$ as a contradiction to our assumption.

(4) $p \in \partial J(u) \Leftrightarrow J(u) + J^*(p) = \langle u, p \rangle \Leftrightarrow J^{**}(u) + J^*(p) = \langle u, p \rangle \Leftrightarrow u \in \partial J^*(p)$. \square

Theorem 1.13. Assume that $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is proper, convex, and lsc. Then J is μ -strongly convex iff J^* is $\frac{1}{\mu}$ -Lipschitz differentiable.

Proof. (only if) Let $p \in \partial J(u)$ be arbitrarily given. By μ -strong convexity of J , we have

$$J(v) \geq J(u) + \langle p, v - u \rangle + \frac{\mu}{2} \|v - u\|^2 \quad \forall v. \quad (1)$$

Then $\forall q : J^*(q) = \sup_v \{\langle q, v \rangle - J(v)\} \leq \sup_v \{\langle q, v \rangle - J(u) - \langle p, v - u \rangle - \frac{\mu}{2} \|v - u\|^2\} = \langle q, u \rangle - J(u) + \sup_v \{\langle q - p, v - u \rangle - \frac{\mu}{2} \|v - u\|^2\} = \langle q, u \rangle - J(u) + \frac{1}{2\mu} \|q - p\|^2 = \langle p, u \rangle - J(u) + \langle q - p, u \rangle + \frac{1}{2\mu} \|q - p\|^2 = J^*(p) + \langle q - p, u \rangle + \frac{1}{2\mu} \|q - p\|^2$. Here we have used the identity $\langle p, u \rangle - J(u) = J^*(p)$. We have actually derived $\lim_{q \rightarrow p} \|J^*(q) - J^*(p) - \langle q - p, u \rangle\| / \|q - p\| = 0$, which asserts that J^* is (Fréchet-)differentiable at p with $\nabla J^*(p) = u$.

Finally we show ∇J^* is $\frac{1}{\mu}$ -Lipschitz. Let $u = \nabla J^*(p)$, $v = \nabla J^*(q)$, or equivalently $p \in \partial J(u)$, $q \in \partial J(v)$. Then by (1) we have

$$\begin{aligned} J(v) &\geq J(u) + \langle p, v - u \rangle + \frac{\mu}{2} \|v - u\|^2, \\ J(u) &\geq J(v) + \langle q, u - v \rangle + \frac{\mu}{2} \|u - v\|^2. \end{aligned}$$

Adding the above two inequalities, we obtain $\mu \|u - v\|^2 \leq \langle p - q, u - v \rangle \leq \|p - q\| \|u - v\|$ and thus $\|u - v\| \leq \frac{1}{\mu} \|p - q\|$.

(if) Note that $J^*(q) = J^*(p) + \int_0^1 \langle \nabla J^*(p + s(q - p)), q - p \rangle ds = J^*(p) + \langle \nabla J^*(p), q - p \rangle + \int_0^1 \langle \nabla J^*(p + s(q - p)) - \nabla J^*(p), q - p \rangle ds \leq J^*(p) + \langle \nabla J^*(p), q - p \rangle + \frac{1}{2\mu} \|q - p\|^2$. Let $p \in \partial J(u) \Leftrightarrow u = \nabla J^*(p)$. Then $J^*(q) \leq J^*(p) + \langle q - p, u \rangle + \frac{1}{2\mu} \|q - p\|^2$. Taking the convex conjugate on both sides, we deduce $J(v) = J^{**}(v) \geq \sup_q \{\langle q, v \rangle - (J^*(p) + \langle q - p, u \rangle + \frac{1}{2\mu} \|q - p\|^2)\} = -J^*(p) + \langle p, v \rangle + \frac{\mu}{2} \|v - u\|^2 = J(u) + \langle p, v - u \rangle + \frac{\mu}{2} \|v - u\|^2$. \square

Theorem 1.14 (weak duality). Let $K \in \mathbb{R}^{m \times n}$, and $F : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$, $G : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ are proper, convex, and lsc. Then it holds that $\inf_u \{F(Ku) + G(u)\} \geq \sup_p \{-G^*(-K^\top p) - F^*(p)\}$.

Proof. Let $\mathcal{L}(u, p) = \langle p, Ku \rangle - F^*(p) + G(u)$, then $\inf_u \{F(Ku) + G(u)\} = \inf_u \sup_p \mathcal{L}(u, p)$ and $\sup_p \{-G^*(-K^\top p) - F^*(p)\} = \sup_p \inf_u \mathcal{L}(u, p)$. It remains to verify $\inf_u \sup_p \mathcal{L}(u, p) \geq \sup_p \inf_u \mathcal{L}(u, p)$. For an arbitrarily fixed (u, p) , we have $\sup_{p'} \mathcal{L}(u, p') \geq \mathcal{L}(u, p) \geq \inf_{u'} \mathcal{L}(u', p)$. Taking infimum over u yields $\inf_u \sup_{p'} \mathcal{L}(u, p') \geq \inf_{u'} \mathcal{L}(u', p)$; Taking supremum over p yields $\sup_{p'} \mathcal{L}(u, p') \geq \sup_p \inf_{u'} \mathcal{L}(u', p)$. Hence, the conclusion follows. \square

Theorem 1.15 (Fenchel-Rockafellar duality). *Assume $\exists \bar{u} \in \text{dom } G$ s.t. F is continuous at $K\bar{u}$. Then the strong duality holds: $\mathcal{P}^* = \mathcal{D}^*$. Moreover, (u^*, p^*) is the optimal primal-dual pair iff*

$$\begin{cases} Ku^* \in \partial F^*(p^*), \\ -K^\top p^* \in \partial G(u^*). \end{cases}$$

Proof. Define $\Phi(v) := \inf_u \{F(Ku + v) + G(u)\}$. Since $\forall v^1, v^2 \in \mathbb{R}^m$, $\alpha \in [0, 1] : \alpha\Phi(v^1) + (1 - \alpha)\Phi(v^2) = \inf_{u^1} \{\alpha F(Ku^1 + v^1) + \alpha G(u^1)\} + \inf_{u^2} \{(1 - \alpha)F(Ku^2 + v^2) + (1 - \alpha)G(u^2)\} = \inf_{u^1, u^2} \{\alpha F(Ku^1 + v^1) + (1 - \alpha)F(Ku^2 + v^2) + \alpha G(u^1) + (1 - \alpha)G(u^2)\} \geq \inf \{F(Ku + \alpha v^1 + (1 - \alpha)v^2) + G(u) : u = \alpha u^1 + (1 - \alpha)u^2\} \geq \Phi(\alpha v^1 + (1 - \alpha)v^2)$, we prove that Φ is a convex function.

Without loss of generality, assume $\Phi(0) > -\infty$. By our assumption, $\exists \epsilon > 0$ s.t. $\forall \|v\| < \epsilon : \Phi(v) \leq F(K\bar{u} + v) + G(\bar{u}) \leq M$ for some $M < \infty$, i.e., $v \in \text{dom } \Phi$. By Theorem 1.2, Φ is locally Lipschitz at 0, and $\Phi(0) = \Phi^{**}(0) = \sup_p -\Phi^*(p)$, where $\Phi^*(p) = \sup_v \{\langle p, v \rangle - \inf_u \{F(Ku + v) + G(u)\}\} = \sup_{v, u} \{\langle p, v + Ku \rangle + \langle -K^\top p, u \rangle - F(Ku + v) - G(u)\} = F^*(p) + G^*(-K^\top p)$. Thus, $\mathcal{P}^* = \mathcal{D}^*$ is proven.

As for the optimality condition, note that $\forall (u, p) : \mathcal{G}(u, p) = F(Ku) + G(u) + G^*(-K^\top p) + F^*(p) = F(Ku) + F^*(p) - \langle Ku, p \rangle + G(u) + G^*(-K^\top p) - \langle -K^\top p, u \rangle \geq 0$. The equality holds, i.e., $\mathcal{G}(u^*, p^*) = 0$, iff $Ku^* \in \partial F^*(p^*)$ and $-K^\top p^* \in \partial G(u^*)$ according to Theorem 1.10. \square

Theorem 1.16 (Moreau identity). *Let $\tau > 0$ and $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be proper, convex, and lsc. Then the following identity holds:*

$$\text{id}(\cdot) = \text{prox}_{\tau J}(\cdot) + \tau \text{prox}_{\frac{1}{\tau} J^*}(\cdot/\tau).$$

Proof. $v = \tau \text{prox}_{\frac{1}{\tau} J^*}(u/\tau) \Leftrightarrow (I + \frac{1}{\tau} \partial J^*)^{-1}(u/\tau) \ni v/\tau \Leftrightarrow \partial J^*(v/\tau) \ni u - v \Leftrightarrow v/\tau \in \partial J(u - v) \Leftrightarrow u - v = (I + \tau \partial J)^{-1}(u) = \text{prox}_{\tau J}(u)$. \square

Theorem 1.17. *Let $F, G : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be proper, convex, and lsc. Then*

$$(F \square G)^* = F^* + G^*.$$

Proof. $\forall p \in \mathbb{E} : (F \square G)^*(p) = \sup_{u, v} \{\langle p, u \rangle - F(v) - G(u - v)\} = \sup_{u, v} \{\langle p, v \rangle - F(v) + \langle p, u - v \rangle - G(u - v)\} = F^*(p) + G^*(p)$. \square

2 Optimization Algorithms

Theorem 2.1. *If $\langle \nabla J(u^k), d^k \rangle < 0$, then $J(u^k + \tau d^k) < J(u^k)$ for all sufficiently small $\tau > 0$.*

Proof. The conclusion follows from Taylor expansion: $J(u^k + \tau d^k) = J(u^k) + \tau \langle \nabla J(u^k), d^k \rangle + o(\tau) = J(u^k) + \tau (\langle \nabla J(u^k), d^k \rangle + o(1)) < J(u^k)$, for all $\tau > 0$ sufficiently small. \square

Lemma 2.2 (feasibility of line search). *Assume that $J : \mathbb{E} \rightarrow \mathbb{R}$ is continuously differentiable, $\langle \nabla J(u^k), d^k \rangle < 0 \forall k$, and $0 < c_1 < c_2 < 1$. Then there exists an open interval in which the step size τ satisfies the Armijo- and the curvature conditions.*

Proof. Consider $\phi(\tau) := J(u^k + \tau d^k)$ and $\psi(\tau) := J(u^k) + \tau c_1 \langle \nabla J(u^k), d^k \rangle$ for $\tau \geq 0$. Since $\phi'(0) = \langle \nabla J(u^k), d^k \rangle < \psi'(0)$, $\phi(\tau) < \psi(\tau)$ for all $\tau > 0$ sufficiently close to 0. On the other hand, $\phi(\cdot)$ is bounded from below but $\psi(\cdot)$ is not. Hence, ϕ and ψ intersect at $\tau = \tau' > 0$ (for the first time as τ increases from 0). Thus, $0 < \tau < \tau'$ fulfills the Armijo condition.

By the mean value theorem, $\exists \tau'' \in (0, \tau')$ s.t. $J(u^k + \tau'' d^k) - J(u^k) = \tau'' \langle \nabla J(u^k + \tau'' d^k), d^k \rangle$. This implies $\langle \nabla J(u^k + \tau'' d^k), d^k \rangle = c_1 \langle \nabla J(u^k), d^k \rangle > c_2 \langle \nabla J(u^k), d^k \rangle$ since $c_1 < c_2$ and $\langle \nabla J(u^k), d^k \rangle < 0$. By continuity, this inequality holds in a neighborhood of τ'' . \square

Theorem 2.3 (Zoutendijk). *Assume that $J : \mathbb{E} \rightarrow \mathbb{R}$ is continuously differentiable, and the Armijo- and curvature-conditions are both satisfied with $0 < c_1 < c_2 < 1$ for each k . In addition, J is μ -Lipschitz differentiable on $\{u \in \mathbb{E} : J(u) \leq J(u^0)\}$. Then $\sum_{k=0}^{\infty} \frac{|\langle \nabla J(u^k), d^k \rangle|^2}{\|d^k\|^2} < \infty$.*

Proof. From the curvature condition, we have $\langle \nabla J(u^{k+1}) - \nabla J(u^k), d^k \rangle \geq (c_2 - 1) \langle \nabla J(u^k), d^k \rangle$. Since ∇J is μ -Lipschitz, $\langle \nabla J(u^{k+1}) - \nabla J(u^k), d^k \rangle \leq \tau^k \mu \|d^k\|^2$. Altogether we have $\tau^k \geq \frac{(c_2 - 1) \langle \nabla J(u^k), d^k \rangle}{\mu \|d^k\|^2}$. Using the Armijo condition, we have $J(u^{k+1}) \leq J(u^k) - \frac{c_1(1 - c_2) |\langle \nabla J(u^k), d^k \rangle|^2}{\mu \|d^k\|^2}$.

Summing up this inequality from $k = 0$ to ∞ , we have $\sum_{k=0}^{\infty} \frac{|\langle \nabla J(u^k), d^k \rangle|^2}{\|d^k\|^2} < \infty$. \square

Lemma 2.4. *Assume that $J : \mathbb{E} \rightarrow \mathbb{R}$ is μ -Lipschitz differentiable. Then $\forall u, v \in \mathbb{E}$:*

$$|J(v) - J(u) - \langle \nabla J(u), v - u \rangle| \leq \frac{\mu}{2} \|v - u\|^2.$$

Proof. Since $J(v) = J(u) + \int_0^1 \langle \nabla J(u + t(v - u)), v - u \rangle dt = J(u) + \langle \nabla J(u), v - u \rangle + \int_0^1 \langle \nabla J(u + t(v - u)) - \nabla J(u), v - u \rangle dt$, we have $|J(v) - J(u) - \langle \nabla J(u), v - u \rangle| = \left| \int_0^1 \langle \nabla J(u + t(v - u)) - \nabla J(u), v - u \rangle dt \right| \leq \int_0^1 |\langle \nabla J(u + t(v - u)) - \nabla J(u), v - u \rangle| dt \leq \int_0^1 \|\nabla J(u + t(v - u)) - \nabla J(u)\| \|v - u\| dt \leq \int_0^1 t \mu \|v - u\|^2 dt = \frac{\mu}{2} \|v - u\|^2$. \square

Theorem 2.5 (convergence of gradient descent). *Assume that $J : \mathbb{E} \rightarrow \mathbb{R}$ is μ -Lipschitz differentiable. Then the gradient descent iteration $u^{k+1} = u^k - \tau \nabla J(u^k)$ with $\tau \in (0, 1/\mu]$ yields $\lim_{k \rightarrow \infty} \nabla J(u^k) = 0$.*

Proof. First, note that $J(u^{k+1}) \leq J(u^k) \forall k$. Since J has finite infimum by assumption, $\lim_{k \rightarrow \infty} |J(u^{k+1}) - J(u^k)| = 0$. Due to the majorization property and $\mu \leq 1/\tau$, we have $J(u^{k+1}) \leq J(u^k) + \langle \nabla J(u^k), u^{k+1} - u^k \rangle + \frac{1}{2\tau} \|u^{k+1} - u^k\|^2 = J(u^k) - \frac{1}{2\tau} \|u^{k+1} - u^k\|^2$. Hence, we conclude $\|\nabla J(u^k)\| = \frac{1}{\tau} \|u^{k+1} - u^k\| \rightarrow 0$. \square