Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 10

Room: 02.09.023

Wednesday, 29.01.2020, 12:15-14:00

Submission deadline: Monday, 27.01.2020, 16:15, Room 02.09.023

Primal-Dual Methods (8+6 Points)

Exercise 1 (4 Points)**.** Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \; g(x) + \sum_{i=1}^{k} f_i(K_i x), \tag{1}$$

with $g : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f_i : \mathbb{R}^{m_i} \to \overline{\mathbb{R}}$ closed, proper, convex and $K_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ linear. Assume that g and all f_i are *simple* in the sense that their proximal mapping

$$\text{prox}_{\tau f_i}(y) := \text{argmin}_{x \in \mathbb{R}^{m_i}} \;\; f_i(x) + \frac{1}{2\tau} \|x - y\|^2 ,$$

can be efficiently computed. Explain how (1) can be solved with PDHG and write down the explicit update equations.
Hint: Stack the individual K_i into a single matrix K.

Exercise 2 (4 Points)**.** Prove that the algorithm

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* \bar{p}^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K u^{k+1}), \\ \bar{p}^{k+1} &= 2p^{k+1} - p^k. \end{aligned} \tag{PDHG*}$$

converges, and the limit of the u^k is a minimizer of $G(u) + F(Ku)$ (with the same assumptions on F, G, and K as in the lecture).

Hint: Show that (PDHG*) is equivalent to an algorithm we discussed in the lecture applied to a reformulated problem!

Exercise 3 (6 Points)**.** Consider following matrix

$$M := \begin{bmatrix} S & -K^\top \\ -K & T \end{bmatrix},$$

where S and T are symmetric matrixes, and $S \succ 0$, $T \succ 0$ (i.e. S and T are positive definite).

- Show that a matrix $A \in \mathbb{R}^{n\times n} \succ 0$ if and only if for an invertible matrix $P \in \mathbb{R}^{n\times n}$, $PAP^\top \succ 0$.

- Show that if $T - KS^{-1}K^\top \succ 0$, then $M \succ 0$.
  Hint: Firstly, manage to compute M^{-1} by solving following equation:

  $$M[u,p]^\top = [x,y]^\top.$$

  To get $T - KS^{-1}K^\top$, you should solve u by x and p. Then substitute u to get p. Secondly, reformulate M^{-1} like:

  $$M^{-1} = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & 0 \\ D & I \end{bmatrix},$$

  where A, B, C, D are four matrixes can be expressed by S, K and T. Finally, using the theorem from first problem to get the conclusion.