Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 3

Room: 02.09.023

Wednesday, 13.11.2018, 12:15-14:00

Submission deadline: Monday, 11.11.2018, 16:15, Room 02.09.023

Subdifferential (10+6 Points)

Exercise 1 (4 Points)**.** Let the convex function $J : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be differentiable at $u \in \text{int}(\text{dom}(J))$. Show that

$$\partial J(u) = \{\nabla J(u)\}.$$

Hint: Use the definition of the subdifferential and the directional derivative. For J being differentiable at the interior of its domain, some direction $v \in \mathbb{R}^n$ and some point $u \in \text{int}(\text{dom}(J))$ the directional derivative $\partial_v J$ of J is given as

$$\partial_v J(u) := \lim_{\epsilon \to 0} \frac{J(u + \epsilon v) - J(u)}{\epsilon} = \lim_{\epsilon \to 0} \frac{J(u) - J(u - \epsilon v)}{\epsilon} = \langle \nabla J(u), v \rangle.$$

Exercise 2 (6 Points)**.** Compute the subdifferential of norms in Euclidean space:

- Let $\|\cdot\|$ be a norm on an Euclidean space $\mathbb{E}$, and $\|\cdot\|_*$ its dual norm defined as

  $$\|p\|_* = \sup_{\|x\| \leq 1} \langle p, x \rangle,$$

  prove that

  $$\partial \|\cdot\| (x) = \{p \in \mathbb{E} : \langle p, x \rangle = \|x\|, \|p\|_* \leq 1\}. \tag{1}$$

  Hint: For $x \neq 0$, we have a generalized Cauchy-Schwarz inequality:

  $$\langle x, y \rangle = \|x\| \langle \frac{x}{\|x\|}, y \rangle \leq \|x\| \cdot \sup_{\|z\| \leq 1} \langle z, y \rangle = \|x\| \|y\|_*, \ \forall x, y \in \mathbb{E}. \tag{2}$$

- Using the result above, compute the subdifferential of the following functions:
  - $J : \mathbb{R}^n \to \mathbb{R}, J(u) = \|u\|_1$.
  - $J : \mathbb{R}^n \to \mathbb{R}, J(u) = \|u\|_2$.
  - $J : \mathbb{R}^n \to \mathbb{R}, J(u) = \|u\|_\infty$.

Exercise 3 (6 points)**.** Given $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, show that the normal cone N_C of the linear-inequality constraints

$$C = \{u \in \mathbb{R}^n : Au \leq b, \} \tag{3}$$

is

$$N_C(u) = \{A^\top \lambda : \lambda \geq 0, \lambda_i = 0 \text{ if } (Au - b)_i < 0\}. \tag{4}$$