Convex Optimization for Machine Learning and Computer Vision

Lecture: Dr. Tao Wu
Exercises: Zhenzhang Ye
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 4

Room: 02.09.023

Wednesday, 20.11.2019, 12:15-14:00

Submission deadline: Monday, 18.11.2019, 16:15, Room 02.09.023

Convex conjugate (14+6 Points)

Exercise 1 (4 points)**.** Let $A \in \mathbb{R}^{n\times n}$ be orthonormal, meaning that $A^\top A = AA^\top = I$. Let the convex set C be given as

$$C := \{u \in \mathbb{R}^n : \|Au\|_\infty \leq 1\}.$$

Compute a formula for the projection onto C given as

$$\Pi_C(v) := \operatorname{argmin}_{u\in\mathbb{R}^n} \frac{1}{2}\|u - v\|_2^2, \quad \text{s.t. } u \in C.$$

Hint: Show that the ℓ_2-norm of a vector is invariant under a multiplication with an orthonormal matrix A, meaning that $\|u\|_2 = \|Au\|_2$.

Exercise 2 (6 points)**.** Assume $J : \mathbb{R}^n \to \mathbb{R}$, compute the convex conjugate of following functions:

- $J(u) = \frac{1}{q}||u||_q^q = \sum_{i=1}^n \frac{1}{q}|u_i|^q$, $q \in [1, +\infty]$. $q = +\infty$ means $J(u) = \|u\|_\infty$.
- $J(u) = \sum_{i=1}^n u_i \log u_i + \delta_{\triangle^{n-1}}(u)$.
- $J(u) = \begin{cases} \frac{1}{2}\|u\|_2^2, & \|u\|_2 \leq \epsilon \\ +\infty, & \text{otherwise} \end{cases}$

Exercise 3 (4 points)**.** Show that projection onto a convex set is Lipschitz continuous with constant equals 1, i.e.

$$||\Pi_C(u) - \Pi_C(v)|| \leq ||u - v||, \ \forall u, v \in \mathbb{E}$$

where C is a convex set.

Exercise 4 (6 points)**.** Let C_i, $1 \leq i \leq n$ be a family of closed convex sets such that

$$\bigcap_{1\leq i\leq n} C_i \neq \emptyset.$$

Show that the problem of finding an element u^* in the intersection

$$u^* \in \bigcap_{1 \le i \le n} C_i$$

can be formulated as the following optimization problem:

$$u^* \in \arg \min_{u \in \bigcap_{i \in \mathcal{I}} C_i} \sum_{\substack{j \notin \mathcal{I} \\ 1 \le j \le n}} d^2(u, C_j),$$

where $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$ can be arbitrary (including the empty set) and $d(z, X)$ is the distance of a point z to the closed convex set X defined as

$$d(z, X) := \min_{x \in X} \|x - z\|_2.$$

Programming: SUDOKU(Due: 25.11) (12 Points)

Exercise 5 (12 Points)**.** Solve the SUDOKU given in the file `exampleSudoku1.mat` (`sudoku array` in python) with projected gradient descent. The algorithm is already given, you only need to figure out the correct formulation.
We represent a SUDOKU as a matrix $\boldsymbol{u} \in \{0,1\}^{9\times 9\times 9}$, where $\boldsymbol{u}_{i,j,k} = 1$ means $\boldsymbol{u}_{i,j} = k$, *i.e.* we fill number k on position (i,j). Therefore, we have following rules, where $f_{i,j}$ are the given entries and B_l is a 9×9 block:

1. Respect given entries: $\boldsymbol{u}_{i,j,k} = 1$, if $f_{i,j} = k$

2. One number for each blank spot: $\sum_k \boldsymbol{u}_{i,j,k} = 1$, $\forall i, j$

3. Numbers occur in a row once: $\sum_j \boldsymbol{u}_{i,j,k} = 1$, $\forall i, k$

4. Numbers occur in a column once: $\sum_i \boldsymbol{u}_{i,j,k} = 1$, $\forall j, k$

5. Numbers occur in a block once: $\sum_{(i,j)\in B_l} \boldsymbol{u}_{i,j,k} = 1$, $\forall B_l, k$

First of all, since the feasible set of u is non-convex, we perform a convex relaxation on it such that $\boldsymbol{u}_{i,j,k} \in [0,1]$.
Since constraints $2-5$ are linear and using the idea from exercise 4, we can vectorize u and try to find a linear operator A, the opimal u^* should satisfy $Au^* = 1$. The problem then can be converted into following convex minimization one:

$$\begin{aligned} u^* &= \operatorname{argmin}_u \frac{1}{2} \|Au - 1\|^2 , \\ s.t.\ & u_{ijk} \in [0,1], \\ & u_{ijk} = 1 \text{ if } f_{ij} = k. \end{aligned}$$

Your task here is try to figure out the linear operator A, such that if u is a valid solution then $Au = 1$.
Hint: 1. How the optimal u^* look like?
2. We vectorize u, therefore figure out the dimension of A first.
3. Starting from the third constraint, how A should look like only for that constraint?