Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 9

Room: 02.09.023

Wednesday, 22.01.2020, 12:15-14:00

Submission deadline: Monday, 20.01.2020, 16:15, Room 02.09.023

Majorize Minimization (12+6 Points)

Exercise 1 (4 Points)**.** Given following convex minimization problem:

$$\min_{x \in \mathbb{R}} f(x) + g(x), \tag{1}$$

where $f(x)$ is non-differentiable and $g(x)$ is L-Lipschitz differentiable, we can apply proximal gradient to solve it.
State that proximal gradient is an example of majorize minimization.
Hint: Write down the explicit updating step of proximal gradient. Then figure out the majorant.

Now consider following theorem:

Theorem. Suppose $J(u)$ is an even, differentiable function on $\mathbb{R}$ such that thte ratio $J'(u)/u$ is decreasing on $(0, \infty)$. Then the quadratic:

$$\hat{J}(v; u) = \frac{J'(u)}{2u}(v^2 - u^2) + J(u)$$

is a majorant of $J(\cdot)$ at the point u.

Exercise 2 (4 Points)**.** Prove above theory.
Hint: 1. Since J is an even function, prove the case $0 \leq v \leq u$ and then generate to other cases.
2. You might use this equation: $J(u) - J(v) = \int_v^u J'(z)dz$.

Exercise 3 (4 Points)**.** The Huber function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given as

$$h_\varepsilon(u) = \begin{cases} \frac{x^2}{2\varepsilon} & \text{if } |x| \leq \varepsilon, \\ |x| - \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

Given the energy function:

$$\text{argmin}_{u \in \mathbb{R}^n} \frac{1}{2} \|u - f\|^2 + H_\varepsilon(Ku), \tag{2}$$

where $f \in \mathbb{R}^n$ is a known variable, the Huber function $H_\varepsilon : \mathbb{R}^m \to \mathbb{R}$ defined as $H_\varepsilon(x) = \sum_{i=1}^m h_\varepsilon(x_i)$ and $K \in \mathbb{R}^{m \times n}$ is a linear operator.

1. State that we can apply above theorem for the Huber function part *i.e.* $H_\varepsilon(Ku)$ (consider the reparameterization).

2. Compute the majorant of the Huber function part using above theorem.

Exercise 4 (6 points)**.** 1. Show that the Huber penalty can be expressed as the infimal convolution of the functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := \frac{x^2}{2\varepsilon}$ and $g : \mathbb{R} \to \mathbb{R}$ with $g(x) := |x|$:

$$h_\varepsilon(x) = (f \,\square\, g)(x).$$

2. Compute the convex conjugate of the function $H_\varepsilon : \mathbb{R}^N \to \mathbb{R}$.

1 Image Denoising (Due Date: 27.01) (10 Points)

Given a noisy input image, we want to remove the noises by solving following minimization problem:

$$\text{argmin}_{u\in\mathbb{R}^N} \frac{1}{2} \|u - f\|^2 + \rho H_\varepsilon(Ku).$$

where $f \in \mathbb{R}^N$ is the input image with N pixels, ρ is a scalar weighting the smooth regularizor, H_ε is the Huber function defined as before and K is the gradient operator.

Your tasks are: (1) use the majorize minimization with majorant in exercise 3 to solve above problem.

(2) check the optimality condition by duality gap with the convex conjugate in exerice 4.