Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 11

Room: 02.09.023
Wednesday, 05.02.2020, 12:15 - 14:00
Submission deadline: Monday, 03.02.2020, 16:15, Room 02.09.023

1 Convergence (6+6 Points)

Exercise 1 (6 Points)**.** Denote $\Pi_C(x)$ as the projection of point x onto a nonempty closed convex set C. Show following properties:

- Π_C is a monotone operator.
- T is firmly nonexpansive, if and only if $||Tx - Ty||^2 \leq \langle x - y, Tx - Ty \rangle, \ \forall x, y$.
- Π_C is firmly nonexpansive.
  Hint: you might use that $\langle y - \Pi_C(x), x - \Pi_C(x) \rangle \leq 0, \forall y \in C$

Solution.

- For any two points u and v, using the definition of projection, we have that $\|u - \Pi_C(u)\|^2 \leq \|u - \Pi_C(v)\|^2$ and $\|v - \Pi_C(v)\|^2 \leq \|v - \Pi_C(u)\|^2$. Summing them up, we have $\|u - \Pi_C(u)\|^2 + \|v - \Pi_C(v)\|^2 \leq \|u - \Pi_C(v)\|^2 + \|v - \Pi_C(u)\|^2$.
  Expanding the squares, we can get the monotonicity of Π_C.
- Recall the proposition of $\frac{1}{2}$-averaged:
  $$\|(I - T)x - (I - T)y\|^2 + \|Tx - Ty\|^2 \leq \|x - y\|^2$$
  Write $\|(I - T)x - (I - T)Y\|^2 = \|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle x - y, Tx - Ty \rangle$, we can achieve the conclusion.
- For two points x and y, using the hint we get $\langle \Pi_C(y) - \Pi_C(x), x - \Pi_C(x) \rangle \leq 0$ and $\langle \Pi_C(x) - \Pi_C(y), y - \Pi_C(y) \rangle \leq 0$. Adding these two yields $\|\Pi_C(x) - \Pi_C(y)\|^2 \leq \langle x - y, \Pi_C(x) - \Pi_C(y) \rangle$. Use the conclusion from second problem, we get the conclusion.

Exercise 2 (6 Points)**.** Prove the Krasnoselskii-Mann Theorem: Let C be a nonempty, closed, convex subset of $\mathbb{E}$, and $u^{k+1} = (1 - \tau^k)u^k + \tau^k \Psi(u^k)$ for $k = 0, 1, 2, ...$ where $\{\tau^k\} \subset [0, 1]$ s.t.

$$\sum_{k=0}^{\infty} \tau^k (1 - \tau^k) = \infty,$$

and $\Psi : C \to C$ satisfies:

1. Ψ is nonexpansive.

2. Ψ has at least one fixed point.

Then $\{u^k\}$ converges to a fixed point of Ψ.

Solution. see lecture notes

TV-ℓ_1 denoising (Due: 10.02)

This time, we have an image with salt-and-pepper noise. Therefore, we use ℓ_1 norm for the data term.

$$\text{argmin}_{u \in \mathbb{R}^N} \|u - f\|_1 + \rho \|Ku\|_1$$

where $f \in \mathbb{R}^N$ is the input image with N pixels, ρ is a scalar weighting the smooth regularizor and K is the gradient operator.
Your task is to apply PDHG to solve above problem.
Hint: for choosing the step size, consider the inequality:

$$\|A\| \leq \sqrt{\max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \max_{1 \leq i \leq m} \sum_{j=1}^{m} |a_{ij}|},$$

where $A \in \mathbb{R}^{m \times n}$.