DAGM 2011 Tutorial on **Convex Optimization for Computer Vision**

Part 1: Convexity and Convex Optimization

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Overview

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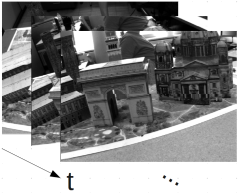
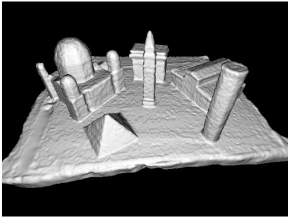
Computer vision deals with inverse problems

- Projection of the 3D world onto the 2D image plane



- Determine unknown model parameters based on observed data



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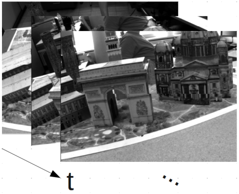

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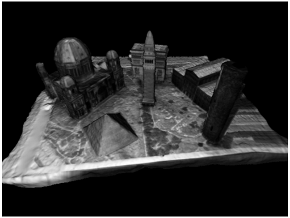

Computer vision is highly ambiguous

What you see ...

Computer vision is highly ambiguous

What you see ... is maybe not what it is!

[Fukuda's Underground Piano Illusion]

Energy minimization methods

- It is in general not possible to solve inverse problems directly
- Add some smoothness assumption to the unknown solution
- Leads to the energy minimization approach

$$\min_u \{E(u) = \mathcal{R}(u) + \mathcal{D}(u, f)\} ,$$

  where f is the input data and u is the unknown solution
- Energy functional is designed such that low-energy states reflect the physical properties of the problem
- Minimizer provides the best (in the sense of the model) solution to the problem

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- Different philosophies:
  - Discrete MRF setting: Images are represented as graphs $\mathcal{G}(\mathcal{V}, \mathcal{E})$, consisting of a node set $\mathcal{V}$, and an edge set $\mathcal{E}$. Each node $v \in \mathcal{V}$ can take a label from a discrete label set $\mathcal{U} \subset \mathbb{Z}$, i.e. $u(v) \in \mathcal{U}$
  - Continuous Variational setting: Images are considered as continuous functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is the image domain.

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- Link to statistics: In a Bayesian setting, the energy relates to the posterior probability via

$$p(u|f) = \frac{1}{Z} exp(-E(u))$$

- Computing the minimizer of $E(u)$ is equivalent to MAP estimation on $p(u|f)$

Example: Total variation based image restoration

Image model: $f = k * u + n$, blur kernel $k =$ 

Variational model: [Rudin, Osher, Fatemi '92]

$$\min_u \int_\Omega |Du| + \frac{\lambda}{2}\|k * u - f\|_2^2$$

(a) Degraded image f

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(a) Degraded image f

(b) Reconstructed image u

Optimization problems are unsolvable

Consider the following general mathematical optimization problem:

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t.}\quad & f_i(x) \le 0\ , \quad i = 1 \dots m \\ & x \in S\ , \end{aligned}$$

where $f_0(x) ... f_m(x)$ are real-valued functions, $x = (x_1, ... x_n)^T \in \mathbb{R}^n$ is a n-dimensional real-valued vector, and S is a subset of $\mathbb{R}^n$

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- Can take more than 30 million years to find an approximate solution!

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[Nesterov '04]

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- Current research: Bridging the gap between convex and non-convex optimization
  - Convex approximations of non-convex models
  - New models
  - Algorithms
  - Bounds

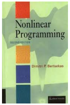
Overview

Literatur on convex analysis and optimization

- Convex optimization, [Boyd,Vandenberghe '04]

- Introductory lectures on convex optimization, [Nesterov '04]

- Nonlinear programming, [Bertsekas '99]

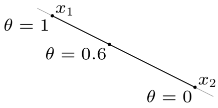


- Variational analysis, [Rockafellar, Wets '88]

Convex sets

- Consider two points x_1, $x_2 \in \mathbb{R}^n$
- The line segment between these two points is given by the points

$$x = \theta x_1 + (1 - \theta) x_2, \; \theta \in [0, 1]$$



Convex sets

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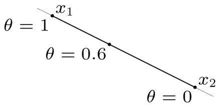
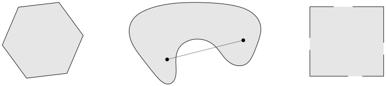


- A set C is said to be convex, if it contains all line segments between any two points in the set

$$x_1,\ x_2 \in C \quad \Rightarrow \quad \theta x_1 + (1-\theta)x_2 \in C,\ \forall\theta \in [0,1]$$

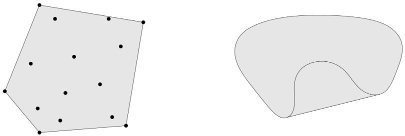


Convex combination and convex hull

- Convex combination of a set of points $\{x_1, ..., x_k\}$, $x_i \in \mathbb{R}^n$ is given by

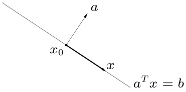
$$x = \sum_{i=1}^{k} \theta_i x_i, \ \theta_i \geq 0, \ \sum_{i=1}^{k} \theta_i = 1$$

- The convex hull of a set of points $\{x_1, ..., x_k\}$ is the set of all convex combinations



A few important convex sets

- Hyperplane: $\{x \mid a^T x = b\}$



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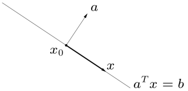
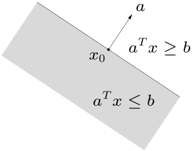


- Halfspace: $\{x \mid a^T x \leq b\}$



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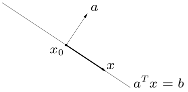

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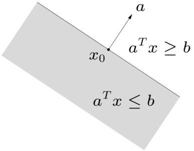
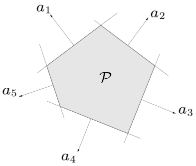


- Polyhedra: Interesection of finitely many hyperplanes and halfspaces



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- Norm Ball: $\{x \mid \|x - x_c\| \leq r\}$, where $\|\cdot\|$ is any norm

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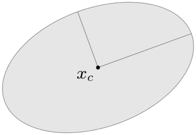
- Ellipsoid: $\{x \mid (x - x_c)P^{-1}(x - x_c) \leq 1\}$, P symmetric and positive definite



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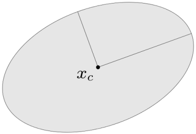
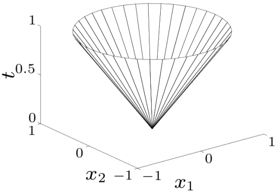


- Norm cone: $\{(x, t) \mid \|x\| \leq t\}$



How to show convexity of a set?

- Check definition

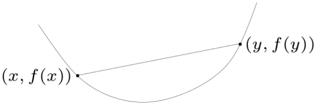
$$x_1,\ x_2 \in C \quad \Rightarrow \quad \theta x_1 + (1-\theta)x_2 \in C,\ \forall \theta \in [0,1]$$

- Show that the set is obtained from simple sets by operations that preserve convexity
  - Intersection
  - Affine transformation (scaling, translation, ...)
  - ...

Convex functions

- Definition: A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, if $\mathrm{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y),\ \forall x, y \in \mathrm{dom}\, f,\ \theta \in [0,1]$$



- f is said to be concave, if $-f$ is convex
- f is strictly convex, if

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y),\ \forall x, y \in \mathrm{dom}\, f,\ x \neq y,\ \theta \in (0,1)$$

Examples of convex functions

Examples on $\mathbb{R}$

- Linear: $ax + b$, $a, b \in \mathbb{R}$
- Exponential: e^{ax}, $a \in R$
- Powers: x^{α}, for $x \geq 0$, $\alpha \geq 1$ or $\alpha \leq 0$
- Powers of absolute values: $|x|^{\alpha}$, $\alpha \geq 0$
- Negative entropy: $x \log x$, for $x \geq 0$

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Examples on $\mathbb{R}^n$

- Affine: $a^T x + b$, $a, b \in \mathbb{R}^n$
- Norms: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$, $p \geq 1$

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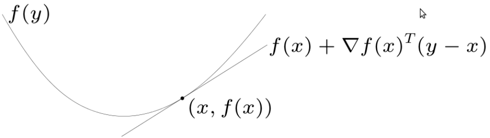
Examples on $\mathbb{R}^{m \times n}$

- Affine: $\mathrm{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}$
- Spectral norm (max non-singular value): $\|X\|_2$

Sufficient conditions of convexity

- **First-order condition:** A differentiable function f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \ \forall x, y \in \mathrm{dom}\ f$$



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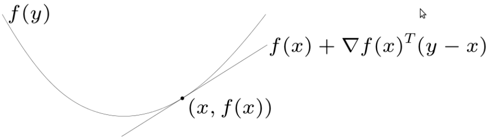

- **Second-order condition:** A twice differentiable function f with convex domain is convex iff

$$\nabla^2 f(x) \succeq 0, \ \forall x \in \operatorname{dom} f$$

- It is strictly convex if the assertion becomes $\nabla^2 f(x) \succ 0$

Example

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- Computing the gradient and Hessian

$$\nabla f(x,y) = \left(\frac{2x}{y}, -\frac{x^2}{y^2}\right), \quad \nabla^2 f(x,y) = \frac{2}{y^3}(y,-x)^T(y,-x)$$

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- Second-order sufficient condition: $\nabla^2 f(x,y) \succeq 0$ for $y \geq 0$

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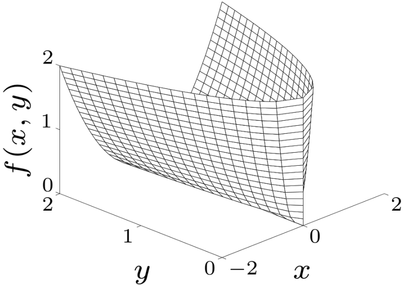

How to verify convexity?

- Verify the definition of convex functions

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y), \ \forall x, y \in \mathrm{dom}\ f, \ \theta \in [0,1]$$

- Sometimes simpler to consider the 1D case

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$$\nabla^2 f \succeq 0 \quad \text{iff} \quad v^T (\nabla^2 f) v \geq 0,\ \forall v \neq 0$$

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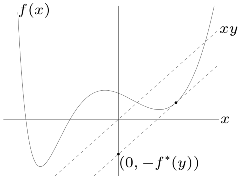
- Show that f is obtained from simple convex functions by operations that preserve convexity
  - Non-negative weihgted sum: $f = \sum_i f_i$, is convex if $\alpha_i > 0$, f_i are convex
  - Composition with an affine function: $f(a^T x + b)$ is convex if f is convex
  - Pointwise maximum: $f(x) = \max\{f_1(x), \ldots, f_n(x)\}$ is convex if $f_1 \ldots f_n$ are convex

The convex conjugate

- The convex conjugate $f^*(y)$ of a function $f(x)$ is defined as

$$f^*(y) = \sup_{x \in \mathrm{dom}\ f} \langle x, y \rangle - f(x)$$



- $f^*(y)$ is a convx function (pointwise supremum over linear functions)
- The biconjugate function $f^{**}(x)$ is the largest convex l.s.c. function below $f(x)$
- If $f(x)$ is a convex, l.s.c. function, $f^{**}(x) = f(x)$

Examples

- $f(x) = |x|$:

$$f^*(y) = \sup_x \langle x, y \rangle - |x| = \begin{cases} 0 & \text{if} \quad |y| \leq 1 \\ \infty & \text{else} \end{cases}$$

- $f(x) = \frac{1}{2} x^T Q x$, Q, posoitive definite

$$f^*(y) = \sup_x \langle x, y \rangle - \frac{1}{2} x^T Q x = \frac{1}{2} y^T Q^{-1} y$$

Duality

- Fenchel's duality theorem

Overview

Convex optimization

- A general convex optimization problem is defined as

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0\,, \quad i = 1 \ldots m \\ & Ax = b \end{array}$$

where $f_0(x) \ldots f_p(x)$ are real-valued convex functions, $x = (x_1, \ldots x_n)^T \in \mathbb{R}^n$ is a m-dimensional real-valued vector, $Ax = b$ are affine equality constraints

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where $f_0(x) \ldots f_p(x)$ are real-valued convex functions, $x = (x_1, \ldots x_n)^T \in \mathbb{R}^n$ is a m-dimensional real-valued vector, $Ax = b$ are affine equality constraints

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- “The great watershed in optimization isn’t between linearity and non-linearity, but convexity and non-convexity” [Rockafellar ’93]

Black box convex optimization

Generic iterative algorithm for convex optimization:

1. Pick any initial vector $x^0 \in \mathbb{R}^n$, set $k = 0$
2. Compute search direction $d^k \in \mathbb{R}^n$
3. Choose step size τ^k such that $f(x^k + \tau^k d^k) < f(x^k)$
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- steepest descend
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- Newton, quasi-Newton
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Black box methods do not exploit the structure of the problem and hence are often less effective

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- Second order cone program (SOCP)

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- Many problems can be formulated in these frameworks
- Fast methods available (simplex, interior point, ...)
- Sometimes ineffective for large-scale problems

Overview

A class of problems

Let us consider the following class of structured convex optimization problems

$$\min_{x \in X} F(Kx) + G(x) \ ,$$

- $K : X \to Y$ is a linear and continuous operator from a Hilbert space X to a Hilbert space Y.
- $F : Y \to \mathbb{R} \cup \{\infty\}$, $G : X \to \mathbb{R} \cup \{\infty\}$ are "simple" convex, proper, l.s.c. functions, and hence have an easy to compute prox operator:

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- It turns out that many standard problems can be cast in this framework.
- There exists a vast literature of numerical algorithms to solve this class of problems

Examples

- Image restoration: The ROF model

$$\min_u \|\nabla u\|_1 + \frac{\lambda}{2}\|k * u - f\|_2^2 \ ,$$

- Compressed sensing: Basis pursuit problem (LASSO)

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- General linear programming problems

$$\min_{x} \langle c, x\rangle \ , \ \text{s.t.} \left\{ \begin{array}{ccc} Ax & = & b \\ x & \geq & 0 \end{array} \right.$$

Primal, dual, primal-dual

The real power of convex optimization comes through duality

Recall the convex conjugate:

$$F^*(y) = \sup_{x \in X} \langle x, y \rangle - F(x) \ , \quad F^{**}(x) = \sup_{y \in Y} \langle x, y \rangle - F^*(y)$$

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$$\max_{y \in Y} -(F^*(y) + G^*(-K^*y)) \quad \text{(Dual)}$$

Allows to compute the duality gap, which is a measure of optimality

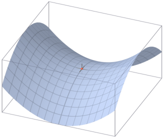
Optimality conditions

We focus on the primal-dual formulation:

$$\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle + G(x) - F^*(y)$$

We assume, there exists a saddle-point $(\hat{x}, \hat{y}) \in X \times Y$ which satisfies the Euler-Lagrange equations

$$\begin{cases} K\hat{x} - \partial F^*(\hat{y}) \ni 0 \\ K^*\hat{y} + \partial G(\hat{x}) \ni 0 \end{cases}$$



Standard first order approaches

1. Classical Arrow-Hurwicz method [Arrow, Hurwicz, Uzawa '58]

$$\begin{cases} y^{n+1} = (I + \tau\partial F^*)^{-1}(y^n + \tau K x^n) \\ x^{n+1} = (I + \tau\partial G)^{-1}(x^n - \tau K^* y^{n+1}) \end{cases}$$

- Alternating forward-backward step in the dual variable and the primal variable
- Convergence under quite restrictive assumptions on τ
- For some problems very fast, e.g. ROF problem using adaptive time steps [Zhu, Chan, '08]

2. Proximal point method [Martinet '70], [Rockafellar '76] for the search of zeros of an operator, i.e. $0 \in T(x)$

$$x^{n+1} = (I + \tau^n T)^{-1}(x^n)$$

- Very simple iteration
- In our framework $T(x) = K^*\partial F(Kx) + \partial G(x)$
- Unfortunately, in most interesting cases, $(I + \tau^n T)^{-1}$ is hard to evaluate
- Hence, the practical interest is limited

Standard first order approaches

3. Douglas-Rachford splitting (DRS) [Mercier, Lions '79]
   - Special case of the proximal point method if the operator T is the sum of two operators, i.e. $T = A + B$

$$\begin{cases} w^{n+1} = (I + \tau A)^{-1}(2x^n - w^n) + w^n - x^n \\ x^{n+1} = (I + \tau B)^{-1}(w^{n+1}) \end{cases}$$

   - Only needs to evaluate the resolvent operator with respect to A and B
   - Let $A = K^* \partial F(K)$ and $B = \partial G$, the DRS algorithm becomes

$$\begin{cases} w^{n+1} = \arg\min_v F(Kv) + \frac{1}{2\tau} \|v - (2x^n - w^n)\|^2 + w^n - x^n \\ x^{n+1} = \arg\min_x G(x) + \frac{1}{2\tau} \left\|x - w^{n+1}\right\|^2 \end{cases}$$

   - Equivalent to the alternating direction method of multipliers (ADMM) [Eckstein, Bertsekas '89]
   - Equivalent to the split-Bregman iteration [Goldstein, Osher '09]
   - Equivalent to an alternating minimization on the augmented Lagrangian formulation

A simple primal-dual algorithm

Proposed in a series of papers: [Pock, Cremers, Bischof, Chambolle, '09], [Chambolle, Pock, '10], [Pock, Chambolle, '11]

- Initialization: Choose s.p.d. T, Σ, $\theta \in [0,1]$, $(x^0, y^0) \in X \times Y$.
- Iterations ($n \geq 0$): Update x^n, y^n as follows:

$$\begin{cases} x^{n+1} &= (I + \mathrm{T}\partial G)^{-1}(x^n - \mathrm{T}K^* y^n) \\ y^{n+1} &= (I + \Sigma\partial F^*)^{-1}(y^n + \Sigma K(x^{n+1} + \theta(x^{n+1} - x^n))) \end{cases}$$

- Alternates gradient descend in x and gradient ascend in y
- Linear extrapolation of iterates of x in the y step
- T, Σ can be seen as preconditioning matrices
- Can be derived from a pre-conditioned DRS splitting algorithm
- Can be seen as a relaxed Arrow-Hurwicz scheme

- The iterations of PD can be written as the variational inequality [He,Yuan '10]

$$\left\langle \begin{pmatrix} x - x^{n+1} \\ y - y^{n+1} \end{pmatrix}, \begin{pmatrix} \partial G(x^{n+1}) + K^* y^{n+1} \\ \partial F^*(y^{n+1}) - K x^{n+1} \end{pmatrix} + M \begin{pmatrix} x^{n+1} - x^n \\ y^{n+1} - y^n \end{pmatrix} \right\rangle \geq 0 \,,$$

$$M = \begin{bmatrix} \mathrm{T}^{-1} & -K^* \\ -\theta K & \Sigma^{-1} \end{bmatrix}$$

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- Convergence is ensured if M is symmetric and positive definite
- This is the case if $\theta = 1$ and the assertion $\|\Sigma^{\frac{1}{2}} K \mathrm{T}^{\frac{1}{2}}\|^2 < 1$ is fulfilled.

A family of diagonal preconditioners

- It is important to choose T, Σ such that the prox-operators are still easy to compute
- Restrict the preconditioning matrices to diagonal matrices
- It turns out that: Let $K \in \mathbb{R}^{m\times n}$, $\mathrm{T} = \text{diag}(\boldsymbol{\tau})$ and $\Sigma = \text{diag}(\boldsymbol{\sigma})$ such that

$$\tau_j = \frac{1}{\sum_{i=1}^{m} |K_{i,j}|^{2-\alpha}}, \quad \sigma_i = \frac{1}{\sum_{j=1}^{n} |K_{i,j}|^{\alpha}}$$

then for any $\alpha \in [0, 2]$

$$\|\Sigma^{\frac{1}{2}} K \mathrm{T}^{\frac{1}{2}}\|^2 \leq 1 \,.$$

- Gives an automatic problem-dependent choice of the primal and dual steps
- Allows to apply the algorithm in a plug-and-play fashion
- For $\alpha = 0$, equivalent to the alternating step method [Eckstein, Bertsekas, '90]

Convergence rates

The algorithm gives different convergence rates on different problem classes [Chambolle, Pock, '10]

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- Coincides with so far best known rates of first-order methods

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The algorithm basically computes matrix-vector products
The matrices are usually very sparse

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Recent GPUs already go into this direction